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Renormalization group operators for maps and universal scaling of universal scaling exponents

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ABSTRACT

Renormalization group (RG) methods provide a unifying framework for understanding critical behaviour, such as transition to chaos in area-preserving maps and other dynamical systems, which have associated with them universal scaling exponents. Recently, de la Llave et al. (2007) [10] have formulated the *Principle of Approximate Combination of Scaling Exponents* (PACSE for short), which relates exponents for different criticalities via their combinatorial properties. The main objective of this paper is to show that certain integrable fixed points of RG operators for area-preserving maps do indeed follow the PACSE.

1. Introduction

A large variety of dynamical systems show scaling behaviour while approaching criticality. The critical properties, such as the scaling exponents for parameter- and phase-space scaling near the critical point, are "universal" in the sense that they are the same for all maps in an open set, within a class such as twist maps, or unimodal maps of an interval, etc. Renormalization group (RG) is a powerful tool for understanding such universal behaviour of critical phenomena in dynamical systems. These techniques have been studied extensively over the past few decades, see e.g. [1,2] for a review and further references. Some of the specific examples of the use of RG are in the studies of the breakup of invariant tori of area-preserving maps [3–5] and Hamiltonian systems [6,7], transition to chaos through period doubling in maps of the interval [8], and many others.

The main idea behind the use of renormalization is the following. A renormalization group operator (RGO) is constructed, so that corresponding to each "universality class," there is a hyperbolic fixed point of this operator, with finitely many unstable directions. The phase- and parameter-space critical exponents appear as the unstable eigenvalues and the phase-space scalings of the RGO at the hyperbolic fixed point. Each of the maps which show the universal critical behaviour belong to the stable manifold of this hyperbolic fixed point.

In many examples, the *rotation number* of the object under consideration, such as the invariant circle, plays a crucial role. The scaling relations discussed in this paper and in, e.g., [3,9] or [10, sec. 2] are for rotation numbers which are quadratic irrationals, though other winding numbers have been studied as well. [11–13] Indeed, the operators discussed below in Section 2 are defined only for quadratic irrational rotation numbers. Any quadratic irrational rotation number ω can be represented [14] by an eventually periodic continued fraction expansion of the form

$$\begin{aligned}\omega &= [0, h_1, \dots, h_l, \overline{p_1, \dots, p_n}] \\ &= [0, h_1, \dots, h_l, p_1, \dots, p_n, p_1, \dots, p_n, p_1, \dots, p_n, \dots] \\ &= \cfrac{1}{h_1 + \cfrac{1}{\dots + \cfrac{1}{h_l + \cfrac{1}{p_1 + \cdots}}}}. \end{aligned} \tag{1}$$

An important property of these universal exponents is that they depend only on the "tail" $P = (p_1, \dots, p_n)$, and are independent of the "head" $H = (h_1, \dots, h_l)$ of the continued fraction expansion. In terms of renormalization, this is understood as follows. The eigenvalues and phase-space scalings of the fixed points of RGO are independent of the head of the continued fraction expansion. This is seen explicitly for the phase-space scalings of the simple twist fixed points and non-twist two-cycles found in [15] for numbers with periodicity $n = 1$. Hence, for the purposes of this paper, it suffices to study the operators and fixed points for numbers of the form

$$\omega_P = [0, \overline{p_1, \dots, p_n}], \tag{2}$$

where of course, not all of the p_i are equal to each other, and cyclic permutations of $(p_1, \dots, p_n)$ are equivalent.

Another important observation regarding these scaling exponents is the recently formulated [10] *Principle of Approximate Combination of Scaling Exponents* (PACSE for short). The main objective

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of this paper is to understand PACSE in terms of renormalization group operators, by studying the simple fixed points of the same form that was already studied in [15]. In the remaining part of this introduction, I will explain the aspects of PACSE relevant to this study. The next section contains the definition of the renormalization group operators and many of the properties of quadratic irrationals which are relevant later. The simple fixed points and two-cycles of these operators, as well as the associated scaling exponents, are derived in Section 3. Section 4 contains some numerical observations regarding scaling of these exponents, while the explicit proof of these scaling properties is presented in Section 5. Some directions for further study will be indicated in the last section, after summarising the main conclusions.

1.1. Principle of approximate combination of scaling exponents

As mentioned above, the scaling exponent for ω of Eq. (1) is the same as that for ω_P of Eq. (2) and is denoted by α_P since it is independent of H. Using the notation of [10], the *concatenation* of two tails P and $Q = (q_1, \dots, q_m)$ is denoted by $PQ = (p_1, \dots, p_n, q_1, \dots, q_m)$. The three parts of PACSE, as formulated in [10], are as follows:

1. There exist constants C_1 and C_2 that depend only on $\max(p_1, \dots, p_n, q_1, \dots, q_m)$, such that
$$C_1 \le \frac{\alpha_{PQ}}{\alpha_P \alpha_Q} \le C_2.$$
2. The following limit exists.
$$\mathcal{A}_\infty = \lim_{k\to\infty} \mathcal{A}_k \quad \text{where } \mathcal{A}_k = \frac{\alpha_{P^kQ}}{(\alpha_P)^k \alpha_Q}.$$
3. The above limit is approached exponentially.
$$|\mathcal{A}_k - \mathcal{A}_\infty| \approx C\xi^k.$$

In this paper, I will calculate the phase-space scaling exponent associated with the simple fixed point of the renormalization group operator for area-preserving maps and show that they follow the above three properties. In particular, we will see later that for these simple fixed points, the last equality is exponentially close to being exact, even for small k and that the constant ξ depends only on P while the constant C depends on Q as well, albeit mildly.

2. Renormalization group operators for quadratic irrationals

In this section, I will recall the renormalization group operators for quadratic irrationals, as defined in [15]. These operators are closely related to those defined earlier in similar contexts in, e.g., [16,11,17]. Below, I will make explicit the form of these operators and some relations between them that will be of relevance to study the universality of the scaling exponents associated with the simple fixed points, as stated in terms of the PACSE.

2.1. Some properties of quadratic irrationals

The convergents of ω_P obtained by truncating the continued fraction expansion in Eq. (2) are

$$\frac{m_i}{n_i} = [0, P, \dots, P, p_1, \dots, p_j], \tag{3}$$

where $j = i \bmod n$ and P is repeated $(i-j)/n$ times. Defining the non-negative integers a_P, b_P, c_P, d_P and the matrix M_P by

$$M_P = \begin{pmatrix} a_P & b_P \\ c_P & d_P \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & p_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & p_n \end{pmatrix}, \tag{4}$$

it is easy to verify that the convergents in Eq. (3) satisfy the following relation.

$$\begin{pmatrix} m_{i+n} \\ n_{i+n} \end{pmatrix} = M_P \begin{pmatrix} m_i \\ n_i \end{pmatrix}.$$

The determinant and trace of M_P are

$$\delta_P = a_P d_P - b_P c_P \quad \text{and} \quad \tau_P = a_P + d_P, \tag{5}$$

and let

$$\Delta_P = \sqrt{\tau_P^2 - 4\delta_P}. \tag{6}$$

A few useful properties of the above matrix are as follows.

1. a_P, b_P, c_P, d_P are simply polynomials in the integers $\{p_1, \dots, p_n\}$ and can be found directly from the following two equations.
$$\frac{a_P}{c_P} = [0, p_1, \dots, p_{n-1}], \qquad \frac{b_P}{d_P} = [0, p_1, \dots, p_n].$$
2. ω_P satisfies the following quadratic equation:
$$c_P x^2 + (d_P - a_P)x - b_P = 0,$$
which has two solutions
$$x = \omega_P = \frac{\tau_P - 2d_P + \Delta_P}{2c_P} \quad \text{and} \quad x = \rho_P = \frac{\tau_P - 2d_P - \Delta_P}{2c_P}.$$
3. From the definition of M_P, it follows that $\delta_P = (-1)^n$. Furthermore, its eigenvalues are
$$\lambda_P^+ = \frac{\tau_P + \Delta_P}{2} = c_P\omega_P + d_P, \quad \lambda_P^- = \frac{\tau_P - \Delta_P}{2} = c_P\rho_P + d_P,$$
with eigenvectors
$$v_P^+ = \begin{pmatrix} \omega_P \\ 1 \end{pmatrix}, \qquad v_P^- = \begin{pmatrix} \rho_P \\ 1 \end{pmatrix}.$$
4. Any sequence of rationals (r_i/t_i) defined by
$$\begin{pmatrix} r_{i+n} \\ r_{i+n} \end{pmatrix} = M_P \begin{pmatrix} r_i \\ r_i \end{pmatrix}$$
using the matrix M_P, converges to ω_P. These form a countable family of sequences of rationals converging to ω_P. It would be interesting to study the usual critical properties of maps using these sequences, instead of only the continued fraction convergents.
5. The matrices $M_p = \begin{pmatrix} 0 & 1 \\ 1 & p \end{pmatrix}$ are precisely those that relate the consecutive continued fraction convergents of the numbers $\omega_p = [0, \overline{p}]$.

The definition of M_P immediately implies the following important relation which will be used later to find relations between the corresponding renormalization group operators. Consider three irrationals $\omega_P = [0, \overline{P}]$, $\omega_Q = [0, \overline{Q}]$, and $\omega_{PQ} = [0, \overline{P, Q}]$, where $P = (p_1, \dots, p_n)$, and $Q = (q_1, \dots, q_m)$ are two distinct tails. Then

$$M_{PQ} = M_P M_Q. \tag{7}$$

2.2. Renormalization group operators

The RGO for area-preserving maps of a cylinder $\mathbb{T} \times \mathbb{R}$ (and also for circle maps) are defined in terms of a pair $U, T : \mathbb{R}^2 \to \mathbb{R}^2$ of area-preserving maps of the covering space $\mathbb{R}^2$ (or the covering space $\mathbb{R}$ for circle maps) that commute with each other; $UT = TU$. For winding numbers of the form ω_P, the RGO $\mathcal{R}_P$ is defined as follows.

$$\mathcal{R}_P \begin{pmatrix} U \\ T \end{pmatrix} = B_P \left[\mathcal{T}_P \begin{pmatrix} U \\ T \end{pmatrix} \right] B_P^{-1} \quad \text{where } \mathcal{T}_P \begin{pmatrix} U \\ T \end{pmatrix} = \begin{pmatrix} U^{a_P} T^{c_P} \\ U^{b_P} T^{d_P} \end{pmatrix}.$$

Note that B_P is different for different fixed points and two-cycles, and in general depends on the orbit of the RGO. Thus B_P is not really a part of the definition of the operator itself. The elements of B_P for a fixed point of $\mathcal{R}_P$ are the phase-space scaling exponents associated with that fixed point. These will be the main objects of study in this paper.

The motivation for defining the operator on the space of commuting pairs on the covering space is the following. The rotation number for the orbits of an area-preserving map $M : \mathbb{T} \times \mathbb{R} \to \mathbb{T} \times \mathbb{R}$ of a cylinder is defined in terms of its lift $\tilde{M} : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ which commutes with translations $T : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ defined by $T(x, y) = (x+1, y)$. E.g., for a period-n orbit of M which satisfies $M^n(x, y) = (x, y)$, the rotation number is m/n if it also satisfies $T^m(\tilde{M}^n(x, y)) = (x, y)$. Thus we consider the commuting map pair $(\tilde{M}, T)$ on the covering space instead of the map M on the cylinder.

From the relation in Eq. (7), it immediately follows that

$$\mathcal{T}_P = \mathcal{T}_{p_n} \circ \cdots \circ \mathcal{T}_{p_1}, \quad \text{and} \quad \mathcal{T}_{PQ} = \mathcal{T}_Q \circ \mathcal{T}_P,$$
$$\text{where } \mathcal{T}_{p_i} \begin{pmatrix} U \\ T \end{pmatrix} = \begin{pmatrix} T \\ UT^{p_i} \end{pmatrix}.$$

3. Simple fixed points and two-cycles of the renormalization

In this section, I will consider the simple (integrable) fixed points and period-two orbits of the renormalization operators studied in the previous section. The main aim will be to find the spatial scaling exponents associated with the fixed point, namely, the diagonal elements of B_P. We will see that the scaling exponents even for these simple fixed points show relations that support the PACSE. The eigenvalues of the RGO at these fixed points are not considered in this paper. Of course, the eventual aim is to find the critical fixed points and the critical scaling exponents. There is already a vast literature about properties of similar RGO, including their critical fixed points, periodic orbits, or the strange attractor set, and the connections with critical phenomena, [1,7,5] but these are not reviewed extensively here since this paper only discusses the simple fixed points.

The simple fixed points I will consider are of the form

$$U_P(x, y) = (x + e_P y + f_P, y), \qquad T_P(x, y) = (x + g_P y + h_P, y), \tag{8}$$

while the two-cycles have the form

$$U_{P\pm}(x, y) = (x + e_{P\pm} y^2 + f_{P\pm}, y),$$
$$T_{P\pm}(x, y) = (x + g_{P\pm} y^2 + h_{P\pm}, y). \tag{9}$$

The spatial scaling will be taken to be diagonal,

$$B_{Pf} = \mathrm{diag}(\alpha_{Pf}, \beta_{Pf}) \quad \text{and} \quad B_{Pc} = \mathrm{diag}(\alpha_{Pc}, \beta_{Pc}), \tag{10}$$

respectively for the fixed point and two-cycle. The main motivation behind this choice is the numerical observations of the critical fixed points of the twist and nontwist maps, which are of the form of Eqs. (8) and (9) respectively. See, e.g., [3,15] for further discussion.

Remark 1. For the map pair of the form

$$U(x, y) = (x + g(y), y), \qquad T(x, y) = (x + h(y), y),$$

all orbits belong to lines $y =$ const, and the winding number $\omega(y)$ of such an orbit and the twist $\tau(y)$ is defined to be

$$\omega(y) = -\frac{h(y)}{g(y)}, \quad \text{and} \quad \tau(y) = \frac{\mathrm{d}}{\mathrm{d}y}\omega(y).$$

The constants $e_P, f_P, g_P, h_P, \alpha_{Pf}, \beta_{Pf}$ and $e_{P\pm}, f_{P\pm}, g_{P\pm}, h_{P\pm}, \alpha_{Pc}, \beta_{Pc}$ in Eqs. (8)–(9) are found using the fact that for the fixed point $\mathcal{R}_P(U_P, T_P) = (U_P, T_P)$, while for the period-two orbit $\mathcal{R}_P(U_{P\pm}, T_{P\mp}) = (U_{P\mp}, T_{P\mp})$. Explicitly, these equations are

$$U_P(x, y) = (B_{Pf} \circ U_P^{a_P} \circ T_P^{c_P} \circ B_{Pf}^{-1})(x, y),$$
$$T_P(x, y) = (B_{Pf} \circ U_P^{b_P} \circ T_P^{d_P} \circ B_{Pf}^{-1})(x, y),$$

and

$$U_{P\mp}(x, y) = (B_{Pc} \circ U_{P\pm}^{a_P} \circ T_{P\pm}^{c_P} \circ B_{Pc}^{-1})(x, y),$$
$$T_{P\mp}(x, y) = (B_{Pc} \circ U_{P\pm}^{b_P} \circ T_{P\pm}^{d_P} \circ B_{Pc}^{-1})(x, y),$$

for all $(x, y) \in \mathbb{R}^2$. The above relations give us linear equations for the constants in Eqs. (8)–(9). These linear equations have a non-trivial solution only if they have a non-zero determinant of the coefficients, which are purely in terms of the constants in Eqs. (10). This condition gives that both $x_{1f} = \alpha_{Pf}$ and $x_{2f} = \alpha_{Pf}/\beta_{Pf}$ are one of the two solutions of the following quadratic equation:

$$\delta_P x_{if}^2 - \tau_P x_{if} + 1 = 0, \quad \text{for } i = 1, 2.$$

This gives four choices for $(\alpha_{Pf}, \beta_{Pf})$. But the only solution which gives a pair (U_P, T_P) with the winding number ω_P at $y = 0$ and a non-zero twist at $y = 0$ is

$$\alpha_{Pf} = \frac{\tau_P + \Delta_P}{2\delta_P} = \frac{\lambda_P^+}{\delta_P}, \quad \text{and}$$
$$\beta_{Pf} = \frac{\tau_P^2 - 2\delta_P + \tau_P \Delta_P}{2\delta_P} = \frac{\alpha_{Pf}^2}{\delta_P}. \tag{11}$$

Using the above, we can get a two-parameter family of fixed points,

$$U_P(x, y) = (x + \mathrm{e}y + f, y), \qquad T_P(x, y) = (x - \rho_P \mathrm{e}y - \omega_P f, y),$$

parametrized by two arbitrary constants (e, f).

Remark 2. It is clear that when $n = 1$, i.e., $P = (p)$, the scaling exponents are the following, and have been known already from [3,15] : $\alpha_{Pf} = -\omega_p$, $\beta_{Pf} = -\omega_p^2$, where

$$\omega_p = [\overline{p}] = \frac{p + \sqrt{p^2 + 4}}{2}. \tag{12}$$

Similarly, for the two-cycle, $x_{1c} = \alpha_{Pc}$ and $x_{2c} = \alpha_{Pc}/\beta_{Pc}^2$ are one of the four solutions of the following quartic equation:

$$\delta_P^2 x_{ic}^4 - (\tau_P^2 - 2\delta_P) x_{ic}^2 + 1 = 0, \quad \text{for } i = 1, 2.$$

This gives sixteen different solutions for the pair $(\alpha_{Pc}, \beta_{Pc})$. For eight of them, the winding number $\omega(y)$ is a constant independent of y – these are the "trivial no-twist" two-cycles. For four out of the remaining eight, the winding number of the $y = 0$ orbit is not equal to ω_P. The only difference between the remaining four is the sign of the terms $e_\pm$ and $f_\pm$. Thus the non-twist two-cycle of interest to us with ω_P as the rotation number of the $y = 0$ orbit has the following scalings.

$$\alpha_{Pc} = \frac{\tau_P + \Delta_P}{2\delta_P}, \quad \text{and} \quad \beta_{Pc} = \pm\frac{\tau_P + \Delta_P}{2} = \pm\frac{\alpha_{Pc}}{\delta_P}.$$

Note that $\alpha_{Pc} = \alpha_{Pf}$ and $\beta_{Pc} = \pm\alpha_{Pf}/\delta_P$. This again gives a two-parameter family of two-cycles,

$$U_P(x, y) = (x \pm \mathrm{e}y^2 + f, y), \qquad T_P(x, y) = (x \pm Key^2 - \omega_P f, y),$$

parametrized by two arbitrary constants (e, f), with $K = [1 - \alpha_{Pc}^2(a_P\tau_P - \delta_P)]/[c_P\alpha_{Pc}^2\tau_P]$.

The above calculations give an explicit expression for the phase-space scaling exponents at the simple fixed points, i.e., the constants $\alpha_{Pf}, \beta_{Pf}, \alpha_{Pc}, \beta_{Pc}$. In order to obtain the parameter-space scalings, we will need to calculate the eigenvalues of the RGO at the above fixed points and two-cycles. Of course, the main objects of interest are such exponents at the critical fixed points, which have not been considered. But we will see that for the above exponents, we can already obtain the observations that support the PACSE. The following two sections will present the relevant results.

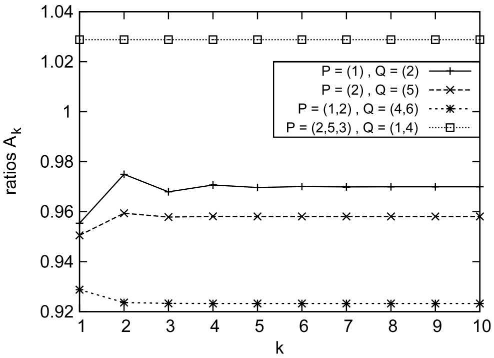

Fig. 1. Plot of $\mathcal{A}_k$ vs. k showing very fast convergence, for various combinations of P and Q.

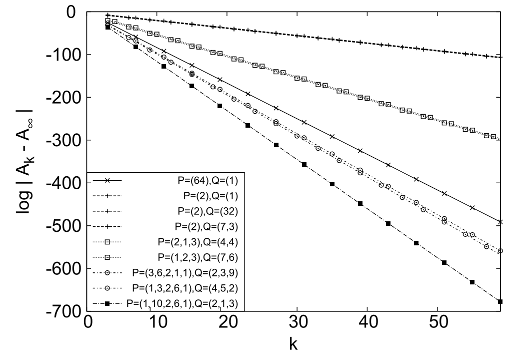

Fig. 2. Plot of $\mathcal{L}_k$ vs. k for various P and Q, showing exponential convergence. A_∞ is taken to be A_{100} in this case.

4. Some numerical observations about the scaling properties of the scaling exponents

I will now consider the relations between the scaling exponents. Since for the simple fixed point, $\beta_{Pf} = \alpha_{Pf}^2/\delta_P$, and $\alpha_{Pc} = \alpha_{Pf} = \pm\delta_P\beta_{Pc}$, it suffices to consider the evidence for PACSE in terms of α_{Pf} alone. Thus let us focus on the sequence

$$\mathcal{A}_k = \frac{\alpha_{(P^kQ)f}}{(\alpha_{Pf})^k(\alpha_{Qf})}, \tag{13}$$

where $P = (p_1, \dots, p_n)$ and $Q = (q_1, \dots, q_m)$. A part of PACSE for this case will state that the above sequence has a limit $\mathcal{A}_\infty$ and the approach to the limit is exponential:

$$|\mathcal{A}_k - \mathcal{A}_\infty| \approx C\xi^k. \tag{14}$$

Thus defining $\mathcal{L}_k = \log|\mathcal{A}_k - \mathcal{A}_\infty|$,

$$\mathcal{L}_k \approx \log(C) + k\log(\xi).$$

This section presents some numerical observations about the quantity $\mathcal{L}_k$ for various choices of P and Q. The main aim of these numerical calculations is simply to gain some intuition regarding the sequence $\mathcal{A}_k$ and they were performed using the arbitrary precision calculator language "bc" available in most Linux systems. The $\mathcal{A}_\infty$ is approximated by $\mathcal{A}_{100}$ in these calculations. Of course, these methods cannot be used to study the critical fixed points, and a much more accurate approach would be needed in that case.

Fig. 1 shows $\mathcal{A}_k$ vs. k for a few representative combinations of P and Q and we can see that the sequences A_k do indeed have limits. The exponential convergence (14) of A_k to a constant A_∞ is shown in Fig. 2, which shows $\mathcal{L}_k$ vs. k for some combinations of P and Q. The slope of this graph is the constant $\log(\xi)$ while the abscissa is the constant $\log(C)$. Fig. 3 shows the plot of $\mathcal{L}_k$ vs. k for fixed $P = (1)$ but different Q. There are several points to be emphasized.

1. ξ depends only on P, and is independent of even the length of Q.
2. ξ does not depend on the order of integers $p_1, \dots, p_n$ within P. This is understood since the exponents α_{Pf} etc., and hence eventually the ratio A_k, depend only on the trace τ_P and the determinant δ_P of M_P. It can be easily established that both δ_P and τ_P are invariant under permutations of these integers. It will be interesting, though numerically fairly challenging, to check whether the same holds true for scaling exponents of the critical (nontrivial) fixed points and for other renormalization group operators.
3. ξ seems to increase in absolute value with $\max(p_1, \dots, p_n)$.
4. C depends on Q, albeit mildly, as seen in the right panel of Fig. 3.

Remark 3. The first part of PACSE states that the ratio $\mathcal{A}_1 = \alpha_{PQ}(\alpha_P\alpha_Q)$ is bounded above and below by constants that depend only on $r_{\max} = \max(p_1, \dots, p_n, q_1, \dots, q_m)$. This seems to indeed be the case for the above exponents and furthermore,

1. C_1 and C_2 do not depend strongly on $r_{\max}$. Indeed, there seem to be constants $C_{\min}$ and $C_{\max}$, independent of P and Q, such that

   $C_{\min} \le \mathcal{A}_1 \le C_{\max}$; and
2. $\mathcal{A}_1$ is very close to one for an overwhelmingly large number of pairs P, Q, i.e.,$C_1 \approx 1$ and $C_2 \approx 1$.

In the integrable case discussed in this paper, $|\mathcal{A}_1|$ is simply the ratio of the largest eigenvalue of the product to the product of the largest eigenvalues of matrices of the form Eq. (4). But the evidence for this observation is not conclusive and I am working on understanding this pattern in detail. It will be instructive to study this also for the critical scaling exponent.

4.1. Scaling of the "exponents" C and ξ

The scaling relation of Eq. (14) is valid only asymptotically in k. The main focus of this section will be to study in detail the approach of the sequence $\mathcal{A}_k$ to the limit $\mathcal{A}_\infty$. In particular, we would like to understand the behaviour of the "constants" ξ and C. For simplicity, I will focus only on numbers with $P = (p)$ and $Q = (q)$. As noted earlier, $\alpha_{Pf} = \omega_p$ and $\alpha_{Qf} = \omega_q$, [see Eq. (12)], and the ratio $\mathcal{A}_k$ is given by

$$\mathcal{A}_k = \frac{\tau_{(P^kQ)} + \Delta_{(P^kQ)}}{2\omega_p^k\omega_q},$$

where $\tau_{(P^kQ)}$ and $\Delta_{(P^kQ)}$ are given in Eq. (5).

Let us define sequences ξ_k and l_k as follows.

$$\xi_k := \frac{\mathcal{A}_{k+1} - \mathcal{A}_k}{\mathcal{A}_k - \mathcal{A}_{k-1}}, \quad \text{and} \quad l_k := \exp[\mathcal{L}_k - k\log(\xi)].$$

Clearly, $\lim_{k\to\infty}\xi_k = \xi$ and $\lim_{k\to\infty} l_k = \log(C)$. In fact, this is how the scaling exponents are usually calculated, since $\mathcal{A}_\infty$ is unknown *a priori*. In parallel with the discussion above, Fig. 4 shows the plot of $\log|\xi_k - \xi|$ and of $\log|l_k - \log(C)|$ vs. k. We thus see that the limits of the above two sequences are also approached exponentially. The plateau in the plot for $\log(C)$ appears because of numerical limitations. We also see that again the slope of this plot, which is a exponent for approach to the limiting value, depends only on P while the abscissa depends on Q as well.

5. The universal scaling of the exponents for the simple fixed points

The main aim of this section is to prove the scaling of the scaling exponents of Eq. (11), i.e., to prove Eq. (14). This is done in a series of steps starting with the relation Eq. (7) in order to calculate the matrix M_{P^kQ} for numbers of the form $\omega_{P^kQ} = [0, \overline{P, \dots, P, Q}]$, where $P = (p_1, \dots, p_n)$ is repeated k times, and $Q = (q_1, \dots, q_m)$, and finally calculating the exponent α_{P^kQf}.

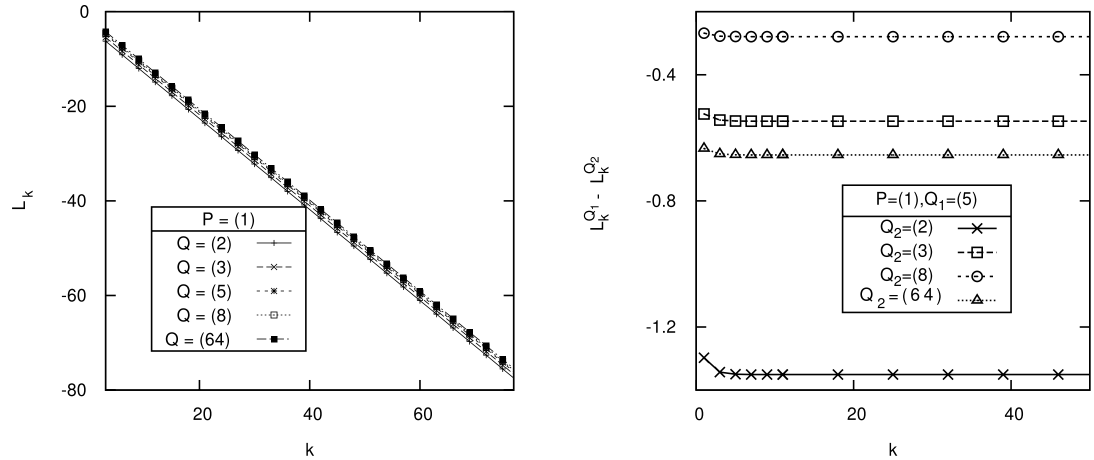

Fig. 3. (Left panel) Plot of $\mathcal{L}_k$ vs. k for fixed $P = \{1\}$ and various $Q = \{2\}, \{3\}, \{5\}, \{8\}$, and $\{64\}$, showing the dependence of $\log(C)$ on Q. (Right panel) $L_k^{Q_1} - L_k^{Q_1}$ vs. k.

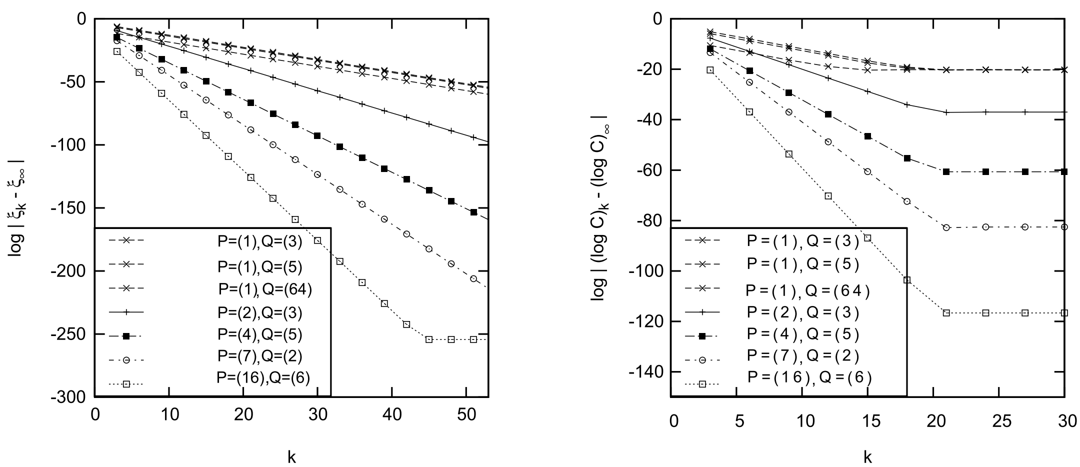

Fig. 4. (Left panel) Plot of $\log|\xi_k - \xi_\infty|$ vs. k; (Right panel) plot of $\log|\ell_k - \log(C)_\infty|$ vs. k. Here ξ_∞ is taken to be ξ_{80} and the same for $\log(C)$.

1. Note that for any matrix, such as M_P of Eq. (4),

   $$(M_P)^k = F_k^{(P)} M_P - \delta_P F_{k-1}^{(P)},$$

   where $F_k^{(P)}$ is a sequence defined by

   $$F_1^{(P)} = 1, \qquad F_2^{(P)} = \tau_P, \quad \text{and} \quad F_{k+1}^{(P)} = \delta_P F_{k-1}^{(P)} + \tau_P F_k^{(P)}.$$

   For example, for $P = (1)$, the $\{F_k^{(1)}\}$ form just the Fibonacci sequence.
2. We can prove by induction, the following closed formula for $F_k^{(P)}$.

   $$F_k^{(P)} = \frac{1}{\Delta_P}\left[\frac{1}{(\alpha_P)^k} - (\alpha_P\delta_P)^k\right], \tag{15}$$

   where α_P is from Eq. (11) (I have dropped the subscript f for simplicity), and δ_P and Δ_P are from Eqs. (5)–(6).
3. The matrix $M_k := M_{P^kQ}$ is then given by

   $$M_k = (M_P)^k M_Q = F_k^{(P)} M_{PQ} - \delta_P F_{k-1}^{(P)} M_Q,$$
   where $M_{PQ} = M_P M_Q$.

   For simplicity of notation, the determinant and trace of the matrix M_k, and the scaling exponent $\alpha_{(P^kQ)f}$, are also denoted with a subscript k instead of P^kQ, and the subscript f for the simple fixed point will be dropped. The determinant and trace of this matrix M_k are

   $$\delta_k = (\delta_P)^k \delta_Q \quad \text{and} \quad \tau_k = \tau_{PQ} F_k^{(P)} - \delta_P \tau_Q F_{k-1}^{(P)}.$$

   Thus, τ_k grows exponentially with k [see the closed form for $F_k^{(P)}$ in Eq. (15)] whereas $|\delta_k| = 1$. Hence, for large k

   $$\Delta_k = \sqrt{\tau_k - 4\delta_k} \approx \tau_k, \quad \text{and hence,} \quad \alpha_k = \frac{\tau_k + \Delta_k}{2\delta_k} \approx \frac{\tau_k}{\delta_k}.$$

   Thus, the scaling exponent for the simple twist fixed point of the RGO for ω_{P^kQ} is given by

   $$\alpha_k \approx \frac{1}{\Delta_P\delta_Q}\left[\frac{\tau_{PQ} - \alpha_P\delta_P\tau_Q}{(\alpha_P\delta_P)^k} + (\alpha_P)^k\left(\frac{\tau_Q}{\alpha_P\delta_P} - \tau_{PQ}\right)\right],$$
   asymptotically for large k.
4. Finally, the ratio $\mathcal{A}_k$ of Eq. (13) is, for large k,

   $$\mathcal{A}_k = \frac{\alpha_{(P^kQ)}}{(\alpha_P)^k(\alpha_Q)} \approx \frac{1}{\Delta_P\delta_Q\alpha_Q}\left[\frac{\tau_{PQ} - \alpha_P\delta_P\tau_Q}{(\alpha_P^2\delta_P)^k} + \left(\frac{\tau_Q}{\alpha_P\delta_P} - \tau_{PQ}\right)\right].$$

   Since $|\alpha_P| > 1$, the first term goes to zero and this gives

   $$\mathcal{A}_\infty = \lim_{k\to\infty} \mathcal{A}_k = \frac{1}{\Delta_P\delta_Q\alpha_Q}\left(\frac{\tau_Q}{\alpha_P\delta_P} - \tau_{PQ}\right).$$

   Thus,

   $$\mathcal{L}_k = \log|\mathcal{A}_k - \mathcal{A}_\infty| \approx \log\frac{|\tau_{PQ} - \alpha_P\delta_P\tau_Q|}{\Delta_P\alpha_Q} + k\log\left(\frac{1}{\alpha_P^2}\right).$$

This completes the proof of the PACSE for the scaling exponents associated with the simple fixed points of the renormalization group operators presented in Sections 2 and 3. From this explicit expression, we can see clearly that the absolute value of the slope of the graphs of $\mathcal{L}_k$ vs. k in Section 4 is simply $2 \log \alpha_P$, which is independent of Q, while the abscissa is the first term in the above expression and does depend on Q.

6. Conclusions and future directions

This paper presented a detailed study of the simple, integrable fixed points and two-cycles of the renormalization group operators (RGO) for area-preserving maps. Explicit expressions for the phase-space scaling exponents associated with these fixed points and two-cycles obtained from this analysis show that they do indeed satisfy the scaling relations formulated in [10] as the *Principle of Approximate Combination of Scaling Exponents* (PACSE for short). In fact the approach of these exponents to the limits proposed in PACSE also shows scaling behaviour. The numerical studies, such as those in [10], of the critical fixed points will show whether this scaling is also valid for these cases.

One of the main directions of future investigations prompted by this study is the calculation of the critical, non-integrable fixed points of these operators in order to support the PACSE. This will involve calculating not only the phase-space exponents, but also the unstable eigenvalues of the RGO linearized at these critical fixed points, since these eigenvalues are the parameter-space exponents. The twist and non-twist fixed points will certainly have vary different properties and it will be interesting to study both of these. It would also be instructive to study numerically the PACSE for critical non-twist maps, though the numerical challenges in this case are far from trivial.

The scaling relations of the exponents obtained in this paper are intimately related to the properties of the periodic continued fraction expansions (CFE) of the quadratic irrationals for which the RGO are constructed. Extending this analysis to other types of renormalization operators (not based on the CFE) will lead to a better and deeper understanding of PACSE in general.

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