Abstract. We consider a one-dimensional model of a compressible fluid in a bounded interval. We want to estimate the density and velocity of the fluid based on the observations for only velocity. We build an observer exploring the symmetries of the fluid dynamics laws. Our main result is that for the linearised system with full observations of the velocity field, we can find an observer which converges to the true state of the system at any desired convergence rate for finitely many but arbitrarily large number of Fourier modes. Our numerical results corroborate the results for the linearised, fully observed system, and also show similar convergence for the full nonlinear system and also for the case when the velocity field is observed only over a subdomain.

Key words. Data assimilation; Observer; Navier-Stokes equation;

AMS subject classifications. 93C20; 93C95; 93B40

1. Introduction. Data assimilation is the problem of estimating the state of a dynamical system described by an evolution equation, typically partial differential equations (PDE), using noisy and partial observations of that system. This has been widely studied in the geophysical context, e.g. meteorology, oceanography, fluid flows, etc.[5, 17, 22]. One of the main tools for this state estimation problem is the analysis of observability, but both mathematical and applied aspects of observability have received only a little attention.[25, 23, 19] An even less studied aspect has been the construction of appropriate observers in the context of data assimilation.

Observers are essentially a modification of the original evolution equation for the system, to incorporate the observations in a feedback term, with the aim that the solution of the observer converges to the solution of the original system being observed. Observers for finite dimensional systems have been well studied in the literature, see, for example [30, 33]. But in many applications such as earth sciences or engineering, the systems are modeled using PDE that are highly nonlinear and, in many instances, chaotic. In such cases of infinite dimensional systems governed by PDE, there are only a few examples available in the literature, mostly for linear systems and very few for nonlinear systems.[20, 29, 32, 11, 13, 14, 16, 31, 24, 6, 28]

Some of the commonly used observers are Kalman Filters or Luenberger observers but the main drawback of these is that they often may break intrinsic properties of the model, e.g. symmetries and/or physical constraints such as balances in geophysical models, see for example [12] and references therein. For nonlinear system possessing certain symmetries, it is natural to seek a correction term which also preserves those symmetries. Such an invariance may make the correction term nonlocal but it may have other desirable properties, for example, independence from the change of coordinates. Recently, there have been attempts to construct, for a variety of systems, observers based on considerations of symmetry,[9, 10, 3]. The observers we construct are motivated by these recent works, as we will see in section 2. Another motivation...
for observer design is the computational cost: a full Kalman filter is usually too expensive for real applications, due to the size of the gain matrices, while an observer might be much more affordable, without degrading the identification process.

The main aim of this work is to develop an observer for a class of PDE inspired by earth system applications, all of which use some approximations of Navier-Stokes equations for fluid flow. In particular, we propose an observer for the nonlinear PDE [equations (2.1)] describing the evolution of a compressible, adiabatic fluid whose velocity field is observed either fully or partially. Our main theoretical result (Proposition 1), supported by extensive numerical results (section 4), is that in the case of complete observations of the velocity, an appropriate choice of parameters leads to convergence of the observer to the true solution at any prescribed rate for arbitrarily large but finite number of Fourier modes.

There has been some work on developing observers for Navier-Stokes based systems. The papers [31, 16, 27] work with finite dimensional approximations of the PDE involved whereas we work with the full PDE itself. A completely different approach based on developing observers using appropriate continuous time limits of discrete time 3D-var or Kalman filter is developed in a series of papers [18, 21, 8] and also in [7, 2, 1, 26, 15, 4]. One of the main contributions of this paper is that we also derive the decay rates for the convergence of the observers and indeed find observers that can decay arbitrarily fast. Since Navier-Stokes based PDE are commonly used in practical data assimilation problems in earth sciences, our work has the potential to be directly relevant to these applications, as we discuss in section 5.

The outline of the paper is as follows. The model and the observer are both introduced in the next section 2. In that section, we also state the linearized version of this problem. We analyze the convergence of this proposed observer for the linearized PDE in section 3 and prove the main result (Proposition 1). We also briefly discuss the difficulties that arise in the theoretical analysis of the nonlinear equation or of the cases with either partial observations or unknown forcing for the linearized equations. In section 4, we present numerical results to substantiate the linear theory. We also present the numerical results showing the efficacy of the observer in estimating the true solution for the partially observed linear system, the observer in the case of unknown forcing term, and the fully nonlinear case. The last section 5 discusses some future directions of research.

2. Compressible Navier-Stokes equations and observers. In this paper, we study a model of compressible fluid in a bounded interval. The density \( \rho(t,x) \) and velocity \( v(t,x) \) of the fluid form the state vector of this model and they obey the following compressible Navier-Stokes system:

\[
\rho_t + (\rho u)_x = 0, \quad (\rho u)_t + (p + \rho u^2)_x = \nu u_{xx},
\]

where the pressure \( p \) is given by the adiabatic equation of state:

\[
p = p(\rho) = \rho^{\gamma},
\]

with \( \gamma \) the adiabatic exponent taken to be 1.4 throughout this paper. We consider solutions over a finite time \([0,T]\) for the space domain \([0,1]\) with periodic boundary conditions:

\[
\rho(t,x) = \rho(t,x+1), \quad u(t,x) = u(t,x+1), \quad \forall \ t \in [0,T], \quad \text{and} \quad \forall \ x \in [0,1],
\]

and similar periodic conditions for all derivatives of \( \rho \) and \( u \).
We assume that the initial conditions \( \rho(0,x) = \rho_I(x) \) and \( u(0,x) = u_I(x) \) are unknown, but we have some information on the solution \( u(t,x) \) of equations (2.1). The goal is to build an observer \((\hat{\rho}, \hat{u})\) for this system, in such a way that both the density and the velocity of the observer \((\hat{\rho}, \hat{u})\) converges towards the solution \((\rho, u)\). We will assume that we have observations of the velocity denoted by \( u(t,x) \) for \( t \in [0,T] \) and \( x \in \Omega \subset [0,1] \) where \( \Omega \) is a subset of \([0,1]\). We will only consider the case \( \Omega = [0, L] \) with \( L \leq 1 \).

### 2.1. Observers.

We introduce the observer \((\hat{\rho}, \hat{u})\), based on the symmetries for the system, satisfying the following set of equations:

\[
\dot{\hat{\rho}} + (\hat{\rho}\hat{u})_x = F_\rho(\hat{\rho}, \hat{u}, u), \quad \text{ (2.4)}
\]

\[
(\dot{\hat{u}}) + (\hat{\rho} + \rho\hat{u}^2)_x = \nu \hat{u}_{xx} + F_u(\hat{\rho}, \hat{u}, u), \quad 0 < t < T, \quad x \in [0,1].
\]

As there are no observations of the density, we first assume that the feedback terms do not depend on \( \hat{\rho} \). Additionally, since these terms should be equal to 0 when \( u \) and \( \hat{u} \) coincide, it is reasonable to consider the following classes of functions:

\[
F_\rho(\hat{\rho}, \hat{u}, u) = \varphi_\rho \ast D_\rho(u - \hat{u}), \quad F_u(\hat{\rho}, \hat{u}, u) = \varphi_u \ast D_u(u - \hat{u}),
\]

where \( D_\rho \) and \( D_u \) are differential (or integral) operators (e.g. \( \partial_x, \partial_t, \ldots \)), and \( \varphi_\rho \) and \( \varphi_u \) are convolution kernels. We assume that these kernels are time independent and hence only functions of \( x \), and we assume that they are isotropic (for 2D or 3D applications). With this choice, the observer preserves the symmetries of the system, since the correction terms are based on a convolution product with an isotropic kernel, making it invariant by rotation or translation.

Note that for the case of partial observation, i.e., when \( \Omega \neq [0,1] \), the feedback terms are present only in \( \Omega \) so that we essentially write the feedback term as:

\[
F_\rho(\hat{\rho}, \hat{u}, u) = \varphi_\rho \ast D_\rho [1_{\Omega}(u - \hat{u})], \quad F_u(\hat{\rho}, \hat{u}, u) = \varphi_u \ast D_u [1_{\Omega}(u - \hat{u})],
\]

Probably, the most simple observer is defined by Luenberger observer (or asymptotic observer): as only \( u \) is observed, then the feedback term is added only in the velocity equation \( (F_\rho = 0) \), and the feedback term is simply \( F_u(\hat{\rho}, \hat{u}, u) = k_u(u - \hat{u}) \), where \( k_u > 0 \) is a constant. We will study more general observers, with the aim at correcting also the density equation with the velocity.

We would like to examine the convergence of the observer solution \((\hat{\rho}, \hat{u})\) to the solution of the observed system \((\rho, u)\) asymptotically in time. In particular, we are interested in the rate of convergence of \( \| \hat{\rho} - \rho \| \) and of \( \| \hat{u} - u \| \) towards zero where we will only consider the \( L^2 \) norm in this paper.

In order to study the theoretical behavior of the observer, we first linearize the system around a steady state of the nonlinear system (2.1) and study the convergence of the observer for the linearized system. We postpone the presentation of numerical results for the nonlinear case to section 4.5.

### 2.2. Stationary solutions.

In order to linearize equations (2.1), let us first look for the stationary solutions: \( \rho_t = 0 \) and \( u_t = 0 \). We write the stationary solution as \( \rho(t,x) = \rho_0(x) \) and \( u(t,x) = u_0(x) \). Then the first of equations (2.1) gives \( \rho_0(x) = C/u_0(x) \) for a constant \( C \) determined by the initial conditions.

The second of equations (2.1) now reads:

\[
(C'u_0^{-\gamma} + C u_0)_x = \nu_0 u_0_{xx} \Leftrightarrow -\gamma C'u_0^{-\gamma-1} u_0 + C u_0 = \nu_0 u_0_{xx}.
\]
Let us multiply this equation by $u_0$ and integrate over the space domain $[0; 1]$:

$$-\gamma C^2 \int_0^1 u_0^\gamma u_{0x} \, dx + C \int_0^1 u_0 u_{0x} \, dx = \nu \int_0^1 u_{0xx} u_0 \, dx.$$  

The two first integrals are both equal to zero, by considering an integration by part, as the boundary conditions are periodic. Then

$$0 = -\nu \int_0^1 (u_{0x})^2 \, dx,$$

meaning that $u_{0x} \equiv 0$, and hence the only stationary solutions of 1D Navier Stokes equation are constants:

$$\rho(t, x) = \rho_0, \ u(t, x) = u_0.$$

### 2.3. Linearization around an equilibrium state.

We now consider an equilibrium state $(\rho_0, u_0)$, and we linearize equations (2.1) around this state:

\begin{align}
\rho_t + u_0 \rho_x + \rho_0 u_x &= 0, \\
\rho_0 u_t + u_0 \rho_t + \gamma \rho_0^\gamma \rho_x + u_0^2 \rho_x + 2 \rho_0 u_0 u_x &= \nu u_{xx},
\end{align}

where only terms linear in $(\rho, u)$ appear. Using (2.8), multiplied by $u_0$, in (2.9), we get:

$$\rho_t + u_0 \rho_x + \rho_0 u_x = 0, \quad \rho_0 u_t + \gamma \rho_0^\gamma \rho_x + \rho_0 u_0 u_x = \nu u_{xx}.$$  

We can rewrite the solution along the characteristics of the equation. The transport coefficient for both density and velocity is $u_0$, and we define

$$\sigma(t, x) = \rho(t, x + u_0 t) \quad \text{and} \quad v(t, x) = u(t, x + u_0 t).$$

Then $\sigma_x(t, x) = \rho_x(t, x + u_0 t)$ and $\sigma_t(t, x) = \rho_t(t, x + u_0 t) + u_0 \rho_x(t, x + u_0 t)$, and similarly for derivatives of $v(t, x)$. Writing equations (2.10) at point $(t, x + u_0 t)$, we get the following set of equations:

$$\sigma_t + \rho_0 v_x = 0, \quad \rho_0 v_t + \gamma \rho_0^\gamma \sigma_x = \nu v_{xx},$$

which is nothing else than equations (2.10) with $u_0 = 0$. So we can now assume that $u_0 = 0$ without any loss of generality, and for simplicity reasons, we set $\rho_0 = 1$. Thus the linearized Navier-Stokes system that we will work with is:

$$\rho_t + u_x = 0, \quad u_t + \gamma \rho_x = \nu u_{xx},$$

with initial conditions $\rho(0, x) = \rho_I(x)$ and $u(0, x) = u_I(x)$, and periodic boundary conditions as in equation (2.3).

As stated above, we assume that the initial conditions $\rho_I(x)$ and $u_I(x)$ are unknown, but we have some information on the solution $u(t, x)$ of equations (2.12). The goal is to build an observer $(\hat{\rho}, \hat{u})$ for this system, in such a way that the observer $(\hat{\rho}, \hat{u})$ converges towards the solution $(\rho, u)$. We will use the same observer as introduced in equations (2.4)-(2.6), except that the left hand side is now linear, just as in equations (2.12):

\begin{align}
(2.13) \hat{\rho}_t + \hat{u}_x &= \varphi_\rho * D_\rho [I_\Omega(u - \hat{u})], \\
\hat{u}_t + \gamma \hat{\rho}_x &= \nu \hat{u}_{xx} + \varphi_u * D_u [I_\Omega(u - \hat{u})],
\end{align}
with periodic boundary conditions, and initial conditions different from the true initial conditions: \( \hat{\rho}(0, x) = \hat{\rho}_I(x) \neq \rho_I(x) \) and \( \hat{u}(0, x) = \hat{u}_I(x) \neq u_I(x) \).

We will now present the main theoretical result about this observer in the linear case, stating that we can choose the feedback terms in order to guarantee any specified rate of convergence for arbitrarily large but finitely many number of Fourier modes. Since we are dealing with linear PDE, we will use Fourier series representation as our main tool.

3. Theoretical study of an observer for 1D linear Navier-Stokes system.

For the case of velocity observations over the full domain \( x \in [0, 1] \), subtracting equations (2.13) from (2.12) (with \( \Omega = [0, 1] \)), we get the following equations for the errors \( r = \hat{\rho} - \rho \) and \( w = \hat{u} - u \):

\[
\begin{align*}
rt + w_x &= -\varphi_\rho * D_\rho w, \\
w_t + \gamma \hat{\rho}_x &= \nu w_{xx} - \varphi_u * D_u w,
\end{align*}
\]

which is exactly identical to equations (2.13), with \((r, w)\) replacing \((\hat{\rho}, \hat{u})\) and with \( u = 0 \). Hence in the rest of this section, we simply work with equations (2.13) with \( u = 0 \), with the understanding that the results really apply to the errors \((r, w)\).

3.1. Damped wave equation formulation. We now eliminate the density in the velocity equation, in order to get an equation for the velocity alone. Starting from (2.13), where feedbacks are given by (2.6), we take the space derivative of the density equation, and the time derivative of the velocity equations:

\[
\begin{align*}
\hat{\rho}_{tx} + \hat{u}_{xx} &= -\varphi_\rho * D_\rho \hat{u}, \\
\hat{u}_{tt} + \gamma \hat{\rho}_x &= \nu \hat{u}_{xxx} - \varphi_u * D_u \hat{u}_t.
\end{align*}
\]

We replace \( \hat{\rho}_{tx} \) in the second equation by its expression given by the first equation, and we obtain:

\[
\begin{align*}
\hat{u}_{tt} - \gamma \hat{u}_{xx} &= \nu \hat{u}_{xxx} + \varphi_u * D_u \hat{u}_t - \gamma \varphi_\rho * D_\rho \hat{u}_x.
\end{align*}
\]

Equation (3.2) is a damped wave equation, with two forcing terms coming from the observer feedbacks. In this case, the goal is to make \( \hat{u} \) converge towards 0. Note that if there is a known forcing term in the linear system, it will also be added in the observer equation, and then, by linearity, it will also disappear.

We now assume that the following differential operators are used:

\[
D_\rho(f) = \partial_x f, \quad D_u(f) = f,
\]

which means that we want the velocity equation to be controlled by the velocity, and the density equation by the space derivative of the velocity. Equation (3.2) now reads:

\[
\begin{align*}
\hat{u}_{tt} - \gamma \hat{u}_{xx} &= \nu \hat{u}_{xxx} - \varphi_u * \hat{u}_t + \gamma \varphi_\rho * \hat{u}_{xx}.
\end{align*}
\]

Remark 1. If we consider the symmetric case, in which only the density \( \rho \) is observed, and not the velocity \( u \), then we obtain the following system:

\[
\begin{align*}
\hat{\rho}_t + \hat{u}_x &= -\psi_\rho * \hat{D}_\rho \hat{\rho}, \\
\hat{u}_t + \gamma \hat{\rho}_x &= \nu \hat{u}_{xx} - \psi_u * \hat{D}_u \hat{\rho}.
\end{align*}
\]

We now want to eliminate \( u \) in order to obtain an equation for the density alone. From the first equation, we can extract \( \hat{u}_x = -\hat{\rho}_t - \psi_\rho * \hat{D}_\rho \hat{\rho} \) and then we need to take the space derivative of the second equation:

\[
\begin{align*}
\hat{u}_{tx} + \gamma \hat{\rho}_{xx} &= \nu \hat{u}_{xxx} - \psi_u * \hat{D}_u \hat{\rho}_x.
\end{align*}
\]
We get then
\[-\dot{\rho}_{tt} - \psi_\rho * \dot{D}_\rho \dot{\rho}_t + \gamma \dot{\rho}_{xx} = -\nu \dot{\rho}_{txx} - \nu \psi_\rho * \dot{D}_\rho \dot{\rho}_{xx} - \psi_u * \dot{D}_u \dot{\rho}_x,\]
or equivalently,
\[(3.5) \quad \dot{\rho}_{tt} - \gamma \dot{\rho}_{xx} = \nu \dot{\rho}_{txx} - \nu \psi_\rho * \dot{D}_\rho \dot{\rho}_{xx} + \psi_u * \dot{D}_u \dot{\rho}_x.\]

By choosing \(\bar{D}_\rho\) equal to the identity operator, and \(\bar{D}_u = \partial_x\), we can then see that we obtain the same system as in (3.4) with \(\varphi_u = \psi_\rho\), and \(\varphi_\rho = (\nu \psi_\rho + \psi_u) / \gamma\). Then all following results also hold in this case.

3.2. Fourier transform. As we are on a periodic space domain \([0; 1]\), we can consider the following Fourier decomposition of the velocity:

\[(3.6) \quad \hat{u}(t, x) = \sum_{k \neq 0} a_k(t) e^{i2k\pi x},\]

where \(a_k(t)\) are the time dependent Fourier coefficients.

Note that if the mean value \(m\) of \(\hat{u}(x)\) is not equal to 0, then the solution converges towards \(\hat{u}(t, x) + m\) instead of \(\hat{u}(t, x)\), as the equation defines the solution up to a constant. Thus, we assume here that the mean value of \(\hat{u}(x)\) is 0 which is equivalent to assuming that the mean values of the initial conditions of the observer \(\hat{u}\) and the solution \(u\) are the same. This does not change the following computations, except for the \(k = 0\) Fourier mode. We denote by \(\varphi_{uk}\) and \(\varphi_{uk}\) the Fourier coefficients of the (time independent) functions \(\varphi_\rho(x)\) and \(\varphi_u(x)\) respectively.

Substituting (3.6) in (3.2), we obtain the following equation for the \(k\)th mode:

\[(3.7) \quad a_k''(t) + \left(\varphi_{uk} + 4\nu k^2 \pi^2\right) a_k'(t) + 4k^2 \pi^2 \gamma (1 + \varphi_{uk}) a_k(t) = 0.\]

3.3. Spectral analysis. Equation (3.7) is a second order differential equation with constant coefficients. The corresponding discriminant is

\[(3.8) \quad \Delta = (\varphi_{uk} + 4\nu k^2 \pi^2)^2 - 16k^2 \pi^2 \gamma (1 + \varphi_{uk}),\]

and the two eigenvalues are

\[(3.9) \quad \lambda_{\pm} = -\frac{\varphi_{uk} + 4\nu k^2 \pi^2}{2} \pm \frac{\sqrt{\Delta}}{2},\]

with the notation \(\sqrt{\Delta} = i\sqrt{-\Delta}\) if the discriminant is negative.

Then the Fourier coefficient of \(\hat{u}\) is given by:

\[(3.10) \quad \text{if } \Delta \geq 0, \quad a_k(t) = C_{k1} e^{\lambda_+ t} + C_{k2} e^{\lambda_- t};\]
\[(3.11) \quad \text{if } \Delta < 0, \quad a_k(t) = C_{k1} \exp\left(-\frac{\varphi_{uk} + 4\nu k^2 \pi^2}{2} t\right) \cos\left(\frac{\sqrt{-\Delta}}{2} t + C_{k2}\right),\]

where \(C_{k1}\) and \(C_{k2}\) are constants given by the initial conditions.

If the discriminant is positive, then the minimal decay rate of the solution is given by the largest eigenvalue \(\lambda_+\). If the discriminant is negative, it is given by the real part of the eigenvalues \(\Re(\lambda_{\pm})\). The imaginary part gives some information about the oscillations in time of the solution. From the equations (2.13), we can also see that the decay rate for \(\dot{\rho}\) is the same as that for \(\hat{u}\).
3.4. Main result. We now give the main result in this framework: for any specified decay rate, we can find convolution kernels $\varphi_\rho$ and $\varphi_u$ such that the observer $\hat{u}$ converges towards $u$ at this specified rate up to any Fourier mode $K$. Indeed, for any mode $k \geq 1$, we can choose appropriate Fourier coefficients of these kernels, but as we will see, their expression does not ensure the convergence of the Fourier series.

**Proposition 3.1.** For any $d > 0$, for any $K > 0$, one can find $(\varphi_{\rho k})_{k \geq 1}$ and $(\varphi_{uk})_{k \geq 1}$ such that the decay rate of the solution $\hat{u}(t, x)$ of equations (2.13) towards $\hat{u} = 0$ is at least $d$ for any Fourier mode $k \leq K$.

**Proof.** Let $d > 0$, $K > 0$, and let $1 \leq k \leq K$. In order to ensure the positivity of all Fourier coefficients of the convolution kernels, from (3.9), we set

$$\varphi_{uk} = \max(0; 2d - 4\nu k^2 \pi^2).$$

For small modes, the diffusion process is not large enough to ensure the decay rate, so we need to add the feedback term. For larger modes, diffusion will be enough, and there is no need to add the feedback (but one can still add a feedback term, and the decay rate will become larger than the specified rate for such modes). Note that even if we drop the $\max$ in (3.12), it still ensures that the decay rate will be at least $d$, but $\varphi_{uk}$ will become negative for $k > \sqrt{\frac{2d}{4\nu \pi^2}}$.

Then, from (3.8), we set

$$\varphi_{\rho k} = \max \left(0; \frac{(\varphi_{uk} + 4\nu k^2 \pi^2)^2}{16 \gamma k^2 \pi^2} - 1 \right).$$

We still consider the positive part, as for small modes and small values of $\varphi_{uk}$, this expression can lead to negative values. The positive part is also theoretically optional, as even for negative $\varphi_{\rho k}$ coefficients, the following results hold.

The above choice of $\varphi_{\rho k}$ ensures that $\Delta \leq 0$, and the decay rate is then exactly $\frac{\varphi_{uk} + 4\nu k^2 \pi^2}{2}$, which is always larger than (or equal to) $d$, because of the choice of $\varphi_{uk}$ in equation (3.12).

The above choice from equations (3.12)-(3.13) can be used for any mode $k \leq K$ which proves the proposition. $\square$

Note that we could use the same definitions from equation (3.13) for $k > K$, but then the Fourier series does not converge, as $\varphi_{\rho k} = O(k^2)$. So we need to truncate the series, and set $\varphi_{\rho k} = 0$ for $k > K$.

For large modes, namely $k > \sqrt{\frac{2d}{4\nu \pi^2}}$, $\varphi_{uk} = 0$, and then there is no issue in considering the inverse Fourier transform and define a convolution kernel $\varphi_u(x)$ with these coefficients.

3.5. Remarks and comparison with the nudging feedback. For large modes, both $\varphi_{\rho k}$ and $\varphi_{uk}$ are set equal to 0. In such modes, the observer is simply a solution of the damped wave equation without any forcing, and the corresponding decay rate $\lambda_+$ of equation (3.7) is equal to $\gamma/\nu$ asymptotically for $k \to \infty$.

We now compare the result presented in the previous section with what can be done with the standard nudging observer, i.e. when $\varphi_\rho \equiv 0$. Let us fix $k$ and define $a = 4\nu k^2 \pi^2$, $b = 4k\pi \sqrt{\gamma}$, and $x = \varphi_{uk}$.

- If $b - a \geq 0$:
  - If $x \geq b - a$, then the discriminant $\Delta$ is positive, and then the maximal decay rate is $-\lambda_+ = -\frac{1}{2} \left[ -(x + a) + \sqrt{(x + a)^2 - b^2} \right]$. This is a
decreasing function of $x$ for positive $x$. So the optimal decay rate is obtained for $x = b - a$, and the optimal rate is then $\frac{a}{2} = 2k\pi\sqrt{\gamma}$.

- If $x < b - a$, then the discriminant is negative, and the decay rate is given by $-\Re(\lambda_\pm) = \frac{x + a}{2}$. So the optimal decay rate is obtained for $x = b - a$, and the optimal rate is the same as in the previous case.

• If $b - a < 0$, then as $x \geq 0$, the optimal rate is reached for $x = 0$, and it is given by $a - \sqrt{a^2 - b^2}$.

So if $k \leq \frac{\sqrt{\gamma}}{\nu\pi}$, then $b - a \geq 0$ and the optimal rate is $2k\pi\sqrt{\gamma}$. If $k > \frac{\sqrt{\gamma}}{\nu\pi}$, the optimal rate is $2\nu k^2 \pi^2 \left(1 - \frac{\nu^2 k^2 \pi^2}{1 - \frac{\gamma}{\nu^2}}\right)$.

The optimal rate increases then with $k$ when $k \leq \frac{\sqrt{\gamma}}{\nu\pi}$, and then decreases for larger values of $k$. The maximum is reached when $k = \frac{\sqrt{\gamma}}{\nu\pi}$, and it is $\frac{2\nu}{\nu}$. At the limit when $k$ goes to infinity, the optimal rate goes to $\frac{2\nu}{\nu}$, so there is no more effect of the feedback, only the diffusion has some impact. In some sense, simple nudging feedback can only increase the decay rate of the very first modes, with an upper bound of $2k\pi\sqrt{\gamma}$, and then, for increasing modes, it can only double the decay rate of diffusion, and even, its action disappears as the mode goes to infinity. This shows that the observer presented in the previous subsection is much better than simple nudging term, as the decay rate can be increased to any arbitrary value.

3.6. Unknown forcing term. We now assume that the original equation has a forcing term in the velocity equation:

\begin{equation}
\rho_t + u_x = 0, \quad u_t + \gamma \rho_x = \nu u_{xx} - f(x, t).
\end{equation}

If this forcing term $f(t, x)$ is known, then we also add it to the velocity equation of the observer (2.13). Then, by considering the difference between the observer and original equations, the forcing term disappears and we are still considering equation (3.4) for the error.

So we now assume that the forcing term is unknown. In this case, we cannot add it inside the observer equation. So the observer equation remains unchanged, and then, the difference between the reference velocity and the observer velocity satisfies equation (3.4) with a forcing term:

\begin{equation}
\dot{u}_t - \gamma \dot{u}_{xx} = \nu \dot{u}_{t} - \nu u_{xx} - \nu u \ast \dot{u}_t + \gamma \nu \rho \ast \dot{u}_{xx} + f_t.
\end{equation}

We now adapt the spectral analysis. We assume that the mean of $f$ is 0 (no bias in the forcing), and then:

\begin{equation}
f(t, x) = \sum_{k \neq 0} b_k(t)e^{2ik\pi x}.
\end{equation}

Then from (3.7), we obtain the following new equation for the $k^{th}$ mode:

\begin{equation}
ap_k''(t) + (\varphi_{uk} + 4\nu k^2 \pi^2) ap_k'(t) + 4k^2 \pi^2 \nu \varphi \rho_k a_k(t) = b_k'(t).
\end{equation}

We just need to find a particular solution to this equation, and add it to the general solution that we found in section 3.3.

We consider a simple case, where the time dependence of the forcing is a sine (or cosine) function:

\begin{equation}
b_k(t) = c_k \sin(2\omega_k \pi t),
\end{equation}

\begin{equation}
\varphi_{uk} + 4\nu k^2 \pi^2 ap_k'(t) + 4k^2 \pi^2 \nu \varphi \rho_k a_k(t) = b_k'(t).
\end{equation}

Then from (3.7), we obtain the following new equation for the $k^{th}$ mode:

\begin{equation}
ap_k''(t) + (\varphi_{uk} + 4\nu k^2 \pi^2) ap_k'(t) + 4k^2 \pi^2 \nu \varphi \rho_k a_k(t) = b_k'(t).
\end{equation}

We just need to find a particular solution to this equation, and add it to the general solution that we found in section 3.3.

We consider a simple case, where the time dependence of the forcing is a sine (or cosine) function:

\begin{equation}
b_k(t) = c_k \sin(2\omega_k \pi t),
\end{equation}

\begin{equation}
\varphi_{uk} + 4\nu k^2 \pi^2 ap_k'(t) + 4k^2 \pi^2 \nu \varphi \rho_k a_k(t) = b_k'(t).
\end{equation}
where $\omega_k$ is the frequency of the forcing oscillation of mode $k$.

Defining $\alpha_k = \varphi_{uk} + 4\nu k^2 \pi^2$, and $\beta_k = 4k^2 \pi^2 \gamma(1 + \varphi_{pk})$, equation (3.16) becomes:

\[(3.18)\quad a''_k(t) + \alpha_k a'_k(t) + \beta_k a_k(t) = 2c_k \omega_k \pi \cos(2\omega_k \pi t).\]

A particular solution is then given by

\[(3.19)\quad a_k(t) = A_k \cos(2\omega_k \pi t) + B_k \sin(2\omega_k \pi t).\]

The constants $A_k$ and $B_k$ are solution of the following linear system:

\[
A_k (\beta_k - 4\omega_k^2 \pi^2) + B_k (2\omega_k \pi \alpha_k) = 2c_k \omega_k \pi,
\]

\[
A_k (-2\omega_k \pi \alpha_k) + B_k (\beta_k - 4\omega_k^2 \pi^2) = 0.
\]

Then, we get:

\[(3.20)\quad A_k = \frac{2c_k \omega_k \pi}{\beta_k - 4\omega_k^2 \pi^2},\]

\[(3.21)\quad B_k = \frac{2\omega_k \pi \alpha_k}{(\beta_k - 4\omega_k^2 \pi^2)^2 + (2\omega_k \pi \alpha_k)^2}.\]

The amplitude of the particular solution is then given by

\[(3.22)\quad D_k = \sqrt{A_k^2 + B_k^2} = \frac{2c_k \omega_k \pi}{\sqrt{(\beta_k - 4\omega_k^2 \pi^2)^2 + (2\omega_k \pi \alpha_k)^2}}.\]

Increasing $\varphi_{pk}$ or $\varphi_{uk}$ (or both) will make $\beta_k$ or $\alpha_k$ (or both) increase, and then $D_k$ will decrease. This means that we can make $D_k$ become as small as we want and the observer will converge towards the true state. But of course, the numerical performance of the observer is severely degraded in comparison with previous cases, as it is usually not possible to consider extremely high values of feedback coefficients from a numerical point of view.

Concerning the density, adapting the previous calculations on the decrease of $\rho$ (knowing the decrease of $u$), we get:

\[(3.23)\quad E_k = \frac{k}{\omega_k} (1 + \varphi_{pk}) D_k,\]

which means that one can adapt the limit amplitude of $\rho$ by changing the values of $\varphi_{pk}$. Theoretically, choosing $\varphi_{pk} = -1$ has the effect of completely removing the influence of the forcing term on the density, but of course, it is not numerically stable or physically consistent to consider negative feedback coefficients.

### 3.7. Partial observations

In the case of partial observations, we define the observation domain $\Omega = [0, L]$ with $L < 1$. In this case, in order to proceed with Fourier analysis, we will need to find the Fourier transform of $\mathbb{I}_\Omega \hat{u}$ because the feedback terms $\varphi \ast \hat{u}$ will be replaced with $\varphi \ast (\mathbb{I}_\Omega u)$. The Fourier series for $\mathbb{I}_\Omega$ is

\[
\mathbb{I}_\Omega = L + \sum_{k=1}^{\infty} \left[ \frac{\sin 2\pi k L}{k} \sin 2\pi k x + \frac{1 - \cos 2\pi k L}{k} \cos 2\pi k x \right].
\]

Thus we see that the Fourier series of $\mathbb{I}_\Omega \hat{u}$ will have Fourier components for all $k$ even in the case when $\hat{u}$ has just a single Fourier mode.
Such coupling of Fourier modes will numerically slightly degrade the performance of the observer, as at any time, some energy will be transferred between different Fourier modes, as in the full nonlinear model.

If we rewrite equation (3.4), we get:

\begin{equation}
\dot{u}_{tt} - \gamma \dot{u}_{xx} = \nu \ddot{u}_{ttxx} + \gamma \varphi \rho \ast (\mathbb{I}_\Omega \dot{u}_{xx}).
\end{equation}

Then, for simplicity reasons, assuming \(\dot{u}\) only has one single Fourier mode \(k\) at a given time, equation (3.7) rewrites:

\begin{equation}
a_k''(t) + (L_{\varphi_{uk}} + 4\nu k^2 \pi^2) a_k'(t) + 4k^2 \pi^2 \gamma (1 + L_{\varphi_{uk}}) a_k(t) = 0.
\end{equation}

Indeed, only the 0th order term \((L)\) in the Fourier decomposition of \(\mathbb{I}_\Omega\) will be kept through the convolution with \(\dot{u}\) (or one of its derivatives). So, the decay rate becomes

\begin{equation}
\frac{\varphi_{uk}L + 4\nu k^2 \pi^2}{2}.
\end{equation}

Of course, this is an approximation, as even if \(\dot{u}\) only has a single Fourier mode at time \(t = 0\), the convolution with a characteristic function leads to mode mixing, as in a nonlinear situation. But we assume here that most of the energy is along mode \(k\) (if only this mode is present at the initial time), which is confirmed by numerical experiments.

4. Numerical experiments on the 1D compressible Navier-Stokes observer. In this section, we report some of the numerical investigations that illustrate the linear theory we discussed above. We also present results of using the same observer as in the linear case for two scenarios for which we have not presented any theoretical results, namely, (i) the observer when the velocity is observed only over a subinterval of the domain in section 4.4, and (ii) in section 4.5, observer for the fully nonlinear equations.

4.1. Numerical configuration. The space domain is \([0; 1]\) and is supposed to be periodic. The discretization involves \(10^2\) grid points, with a step \(\Delta x = 10^{-2}\). We consider a time step \(\Delta t = 10^{-3}\). We also experimented with increasing the spatial resolution (and correspondingly decreasing the time step), but the results are almost identical and not presented here. The adiabatic exponent is set to \(\gamma = 1.4\), and the diffusion is set to \(\nu = 5 \times 10^{-2}\) (except in section 4.5). The numerical code uses a conservation form of the incompressible Navier-Stokes system, with \(\rho\) and \(\rho u\) as variables. A finite volume scheme is used, in which the inviscid flux is computed using an approximate Riemann solver (e.g. VFRoe scheme). Time integration scheme is a third order explicit Runge-Kutta scheme, where the time step is chosen based on a CFL condition.

In order to reproduce a quasi-linear situation, we consider the true (observed) solution \(\rho(t,x) = 1\) and \(u(t,x) = 0\) while the initial conditions for the observer are set to:

\begin{equation}
\dot{\rho}_I(x) = 1 + 5 \times 10^{-2} \sin(2k\pi kx), \quad \dot{u}_I(x) = 5 \times 10^{-2} \sin(2k\pi kx),
\end{equation}

so that the mean values of \(\dot{\rho}\) and \(\dot{u}\) are \(\rho_I = 1\) and \(u_I = 0\) respectively, and where \(k\) is a given mode, usually the first one \((k = 1,\) unless differently specified).
Fig. 1. The $L^2$ norm of the difference between the observer $(\hat{\rho}, \hat{u})$ and the solution $(\rho, u)$ versus time. Solid and dotted lines are the errors in $\rho$ and $u$, respectively. The left panel is for fixed $\varphi_\rho = 0$ with varying $\varphi_u$, while the right panel is for fixed $\varphi_u = 20$ with varying $\varphi_\rho$.

We first look at the solution without any feedback term. Figure 1 (solid and dashed curves, left panel) show the evolution (in log scale) of the $L^2$ norm of the difference between the observer $(\hat{\rho}, \hat{u})$ and the solution $(\rho, u)$. As there is no feedback, all the Fourier coefficients $\varphi_{\rho k}$ and $\varphi_{u k}$ are equal to 0, and then from equation (3.8), the discriminant is $\Delta \approx -217.2$, and the theoretical decay rate (only due to diffusion) is given by equation (3.9): $d_{th} = 0.987$. Also the oscillation period can be computed from (3.11): $\omega_{th} = \frac{4\pi}{\sqrt{\gamma}} \approx 0.85$.

Numerically, the slope of the (red) curve in the left panel of figure 1 gives a numerical decay rate $d_{num} = 0.980$. Note that the figures show the errors to base 10, hence the slope of the semilog plot is $d_{num}/\log(10) = 0.426$. The numerical oscillation period is approximately $\omega_{num} = 0.852$. Note that one period corresponds to two oscillations on the figure for the norm of the cosine. This excellent agreement between theoretical and numerical values can be reproduced for other modes and other values of the parameters.

4.2. Simple nudging observer. We now suppose that there is some feedback only in the velocity equation. We then first let $\varphi_\rho = 0$, and only modify the values of $\varphi_u$. This simulates the nudging, or asymptotic observer: as only the velocity is measured, only the velocity is corrected in the observer system. Table 1 shows the theoretical and numerical decay rates and oscillation periods for several values of $\varphi_u$.

The first remark is that the numerical results perfectly match the theoretical results, except for the particular value of $\varphi_u = 12.895$. In this case, the numerical decay rate is slightly smaller than the theoretical one. Also, no oscillations can be seen on the results, which is reasonably in agreement with a theoretical period of 81 which will be impossible to see with a final time of $T = 5$.

As $\varphi_u$ increases, the decay rate increases, until $\varphi_u$ reaches $4\pi\sqrt{\gamma} - 4\nu\pi^2 \approx 12.895$ (see remark in section 3.5), for which the discriminant is equal to 0, and then positive for increasing $\varphi_u$. The corresponding optimal decay rate is $d_{th} = 2\pi\sqrt{\gamma} \approx 7.434$ (see section 3.5). We can see that the decay rate then decreases, as the discriminant takes larger positive values, so that one of the two eigenvalues gets closer to 0.

One can see on figure 1 (left panel, blue and magenta curves) that the error decreases much stronger than the case of no feedback (figure 1, left panel, red curve). We can also see that with increasing $\varphi_u$, the period of oscillations increases and eventually there are no oscillations (discriminant is positive). It confirms that the
The theoretical and numerical decay rates and oscillation periods for several values of $\varphi_u$ (k = 1, $\varphi_\rho = 0$). The values in parenthesis give the numerical decay rates in base-10, in order to compare with slopes of lines in figure 1.

<table>
<thead>
<tr>
<th>$\varphi_u$</th>
<th>Theoretical decay rate</th>
<th>Numerical decay rate</th>
<th>Theoretical oscillation period</th>
<th>Numerical oscillation period</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.987</td>
<td>0.980(0.426)</td>
<td>0.85</td>
<td>0.86</td>
</tr>
<tr>
<td>0.1</td>
<td>1.037</td>
<td>1.032</td>
<td>0.85</td>
<td>0.85</td>
</tr>
<tr>
<td>0.5</td>
<td>1.237</td>
<td>1.237</td>
<td>0.86</td>
<td>0.85</td>
</tr>
<tr>
<td>1</td>
<td>1.487</td>
<td>1.486</td>
<td>0.86</td>
<td>0.87</td>
</tr>
<tr>
<td>5</td>
<td>3.487</td>
<td>3.485</td>
<td>0.97</td>
<td>0.98</td>
</tr>
<tr>
<td>10</td>
<td>5.987</td>
<td>6.012(2.61)</td>
<td>1.42</td>
<td>1.38</td>
</tr>
<tr>
<td>12.895</td>
<td>7.434</td>
<td>6.590</td>
<td>81.0</td>
<td>−</td>
</tr>
<tr>
<td>15</td>
<td>4.393</td>
<td>4.364(1.895)</td>
<td>−</td>
<td>−</td>
</tr>
<tr>
<td>20</td>
<td>2.897</td>
<td>2.861</td>
<td>−</td>
<td>−</td>
</tr>
</tbody>
</table>

Table 1

Theoretical and numerical decay rates and oscillation periods for several values of $\varphi_\rho$ with fixed $\varphi_u = 20$ for the $k = 1$ mode. (The values in parenthesis are again decay rates in base-10 for comparison with figure 1.)

<table>
<thead>
<tr>
<th>$\varphi_\rho$</th>
<th>Theoretical decay rate</th>
<th>Numerical decay rate</th>
<th>Theoretical oscillation period</th>
<th>Numerical oscillation period</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2.897</td>
<td>2.861</td>
<td>−</td>
<td>−</td>
</tr>
<tr>
<td>0.5</td>
<td>4.838</td>
<td>4.790(2.080)</td>
<td>−</td>
<td>−</td>
</tr>
<tr>
<td>1.184</td>
<td>10.94</td>
<td>9.74</td>
<td>−</td>
<td>−</td>
</tr>
<tr>
<td>1.5</td>
<td>10.98</td>
<td>11.03(4.791)</td>
<td>1.50</td>
<td>1.57</td>
</tr>
<tr>
<td>5</td>
<td>10.99</td>
<td>11.03</td>
<td>0.43</td>
<td>0.43</td>
</tr>
<tr>
<td>10</td>
<td>10.99</td>
<td>11.06(4.802)</td>
<td>0.28</td>
<td>0.28</td>
</tr>
</tbody>
</table>

Table 2

decay rate decreases if $\varphi_u$ is increased too much.

Similar results have been observed for other values of the diffusion $\nu$, and for other modes $k$, and in each case, these numerical results match well with the theoretical predictions.

4.3. Results on the full observer. We now use the full observer, with an additional feedback term in the density equation. We first set $\varphi_u = 20$, for which the discriminant is positive, the largest eigenvalue gets closer to 0, and then the decay rate becomes non optimal.

Table 2 shows the theoretical and numerical decay rates (and oscillation periods) for several values of $\varphi_\rho$. As the discriminant is positive when $\varphi_u = 20$ and $\varphi_\rho = 0$, equation (3.13) gives the theoretical value for which the discriminant comes back to negative values: $\varphi_\rho = \frac{(\varphi_u + 4\nu^2\pi^2)}{16\nu^2} - 1 \simeq 1.184$.

We observe numerically that the decay rate is very similar to the theoretical rate, and we clearly observe the transition from positive to negative discriminant with the stabilization of the decay rate, and the apparition of oscillations. Increasing $\varphi_\rho$ to a much larger value than the optimum given by (3.13) is not necessary, as the decay rate does not increase, and the period of oscillations increases quite quickly. This is clearly seen from the plots in the right panel of figure 1.

As we have seen, by adding the derivative of the velocity as a feedback to the
density equation, we were able to significantly increase the decay rate of the error, in comparison with only a feedback in the velocity equation.

4.4. Observers with partial observations. In this section we present the numerical results of using observer with feedback term in equation (2.6). The left panels of figure 2 shows the decay rate of the $L^2$ error between the observer $(\hat{\rho}, \hat{u})$ and the actual solution $(\rho, u)$.

We see that, as expected, the rate of decay is smaller for smaller observational intervals. We also see that the error in velocity decreases linearly (with oscillations) whereas the error in density saturates at a fairly high but constant value. This is because the solution of the observer equation converges to a solution with $\hat{u} = 0$ but with $\hat{\rho} = \rho_n$ where $\rho_n \neq \rho_I$ - i.e. the observer density is shifted by an amount which compensates for the initial discrepancy between the mean of $\hat{\rho}$ and the mean of $\rho$. This is not surprising since the original equations themselves are invariant under the constant shift in density.

In order to overcome this problem in the case of partial observations, we propose the following modification of the feedback terms in equation (2.6):

$$F_\rho(\hat{\rho}, \hat{u}, u) = \varphi_\rho * D_\rho [1_{\Omega}(u - \hat{u}) - \langle u - \hat{u} \rangle],$$

where $\langle f \rangle$ indicates average of $f$ over the interval $[0, L]$. This ensures that the average of the feedback term is zero and hence the equilibrium solution of this equation also has mean zero. Note that in the case of $L = 1$, i.e., the case of full observations,
Fig. 3. The decay rates for the velocity and the density, as a function of the length of the observation interval (quasi-linear case in the left panel and fully nonlinear case in the right panel). Note that the slope of the best fit line $\approx 4.5$ whereas the slope of best fit line by taking only the points for $L = 0.1, 0.2, 0.3, 0.4, 0.9, 1.0$ is $\approx 5.7$.

this average is just zero and the feedback in equation (4.2) is identical to that in equation (2.6). The errors obtained by using this new observer are shown in the right upper panel of figure 2. We see clearly that the observer $\hat{\rho}$ now approaches the solution $\rho$ and the error decreases as expected.

In order to clearly see the effect of the observer, the lower right panel of figure 2 shows the actual observer solutions for the case when $\Omega = [0, 0.3]$. They clearly show the effect of incorporating the observations, and also the difference between the observer with the incorrect mean and the one with correct mean. The effect is of course more pronounced on the density than on the velocity.

In this case the discriminant of the decay rate is clearly negative (as evidenced by oscillations in the plot for errors) but we can calculate the decay rate. Figure 3 shows the decay rate of the observer as a function of the length of the interval over which the velocity is observed, for the case when $\varphi_0 = 0.5, \varphi_u = 10$ for the $k = 1$ mode. We notice that this is pretty close to a straight line and the best fit line is given as follows:

$$D = 4.99L + 0.24.$$  

Comparing this with the rate given in equation 3.26, we see that the last equality is a reasonable approximation. Thus even though we cannot calculate the exact decay rate (see section 3.7), a reasonable estimate can be obtained by using formula 3.26:

$$D \approx \frac{\varphi_u L + 4\nu\pi^2 k^2}{2} = 5L + 0.987$$  

in experiments corresponding to figure 3 ($\varphi_u = 10$).

This figure shows that the global slope of the numerical decay rates versus the length of the observation interval is close to $\frac{\varphi_u}{2} = 5$. There is also a good agreement between numerical decay rates and the approximated theoretical ones (see equation 3.26) when $L$ is either small (almost no observations) or large (almost all the domain is observed), while the numerical decay rates are degraded when only half the domain is observed. When $L \approx 0.5$, previous spectral studies show that mixing of Fourier modes has a higher effect than when $L \approx 0$ or 1. Convolution with the characteristic function leads to non negligible transfers of energy from mode $k$ to other modes, and then to a smaller decay rate.
4.5. Observer for nonlinear Navier-Stokes system. In this section, we will report the numerical results of using the observer for the nonlinear system of equations, i.e., from equations (2.4), with the feedback terms as in equations (2.6). Note that we do not have theoretical estimates of the decay rates towards the equilibrium solution. The aim of this section is to understand the efficacy of the above nonlinear observer.

Firstly, we consider observations of the equilibrium solution, so that we are essentially studying the decay of the observer towards the equilibrium. The left panel of figure 4 shows the decay rates of the observer solution with the following two initial conditions, with the feedback term set to zero, and for the case with $(\varphi_\rho, \varphi_u) = (0.2, 10)$:

$$\hat{\rho}_I(x) = 1 + 5 \times 10^{-1} \sin(2k\pi kx), \hat{u}_I(x) = 5 \times 10^{-1} \sin(2k\pi kx),$$

Note that the perturbation strength is 10 times larger than the “quasi-linear” case with the perturbation in equation (4.1). We consider two cases, $k = 1$ and $k = 3$. The cases with other values of $k > 1$ show behaviour very similar to the case $k = 3$ and hence is not shown here explicitly. For reference, we have also plotted the decay rates of the linear case as well.

The right panel of that figure shows the actual observer solutions for the $k = 1$ case. Note the different scales for the nonlinear (left axis) and linear (right axis) regimes. We clearly see that with perturbation strength of $5 \times 10^{-1}$, even though the initial condition only has $k = 1$ mode present, at time $t = 0.1$, higher modes are excited (dotted line, with no feedback). When the observer feedback is added, the solution very quickly decays to before higher modes are excited, bringing the perturbation to
the level where the linear theory is a good approximation and thus the decay rate is identical to that of the linear case of perturbation strength $5 \cdot 10^{-2}$.

The case for $k = 3$ (and indeed all higher modes with $k > 1$) is quite interesting. In this case, the initial condition of $k = 3$ excites the $k = 1$ mode. Thus even with the observer feedback is added, the decay rate is not the same as the linear $k = 3$ decay rate but rather it is closer to the linear $k = 1$ rate. The same behavior is seen for other modes with $k > 1$. Figure 4 shows example of this decay for $k = 3, 6$, along with the actual observers. It is quite clear that in the quasi-linear case, modes other than the one contained in the initial condition are not excited while in the nonlinear case, they are quite clearly excited.

Finally, we also performed numerical experiments with fully nonlinear observer with partial observations over various domain sizes, as discussed in detail in section 3.7. Surprisingly, the behavior in the nonlinear case is very similar to the linear case: the decay rate is close to being linear in the size of the domain - see the right panel of figure 3 and the bottom left panel of figure 4.

5. Conclusion. In this paper, we were interested in observer design for a viscous Navier-Stokes equation. Thanks to intrinsic properties of the system (symmetries), we were able to design observers in order to reconstruct the full solution (both velocity and density) when only one variable (either velocity or density) is observed, even partially.

A spectral study showed that for a tangent linear system (linearized around an equilibrium state), we can prove the convergence of the observer towards the solution, and we can control the decay rate of the error, with explicit formulas for the feedback coefficients as functions of the desired decay rate.

Numerical experiments are in perfect agreement with theory in the linearized situation: we can obtain any decay rate by increasing the observer coefficients. Numerical experiments also show that our observer is still very efficient in the case of partial observations, and also in the full nonlinear case.

The application of this kind of observer in other specific cases such as two or three dimensional Navier-Stokes will be a natural extension of this work, as will be the case of full primitive equations of the ocean or the atmosphere. Observers for fluid equations coupled to other quantities such as temperature or salinity with observations of these fields instead of velocity observations will also be an interesting extension that will be of great interest in practice since these types of measurements are more common. It will also be quite challenging and interesting for practical applications to consider observers in the case when observations are available discretely in time and/or in space.

REFERENCES


