the gravitational-wave memory

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Outline of this talk:

• What is the memory and why is it interesting?

• How do we calculate the memory?

• Is it observable?
Examples of memory:

Two-body scattering/hyperbolic orbits

[ Turner ‘77, Turner & Will ‘78, MF ‘11 ]
Examples of memory: Core-collapse supernovae

[ Burrows & Hayes ’96 ]

[Murphy, Ott, & Burrows ’09]
Examples of memory:

Binary black-hole mergers

The memory slowly builds up during the inspiral, grows rapidly during the merger, and saturates to its final value during the ringdown.
Why is this called “memory”?

[ GW propagating perpendicular to the screen ]
Why is this interesting?

The nonlinear memory is unique among the many other nonlinear effects present in the gravitational-wave signal:

- it is non-oscillatory and visually distinctive in the waveform.
- it has a clear interpretation in terms of the GWs produced by GWs (more later).

像GW“尾巴”，非线性记忆是**hereditary**：

非线性记忆依赖于信息**on**和**inside**过去的光锥。
Why is this interesting?

The nonlinear memory is unique among the many other nonlinear effects present in the gravitational-wave signal:

• Unlike other post-Newtonian corrections, the memory affects the waveform at leading (Newtonian) order. Its “hereditary” nature allows a small effect to build-up to a large value over time.

\[ h_+ = -2 \frac{\mu}{R} v_{\text{orb}}^2(t) \left[ (1 + \cos^2 \iota) \cos[2\varphi(t) - 2\Phi] + \frac{1}{96} \sin^2 \iota(17 + \cos^2 \iota) + O(v_{\text{orb}}) \right] \]

[ Wiseman & Will ’91 ]

• The nonlinear memory is observable and could serve as a **test of general relativity**.
Understanding the memory: the linear memory effect

\[ M \quad \mu \]

\[ x_j(t) \]

\[ \dot{x}_j(t) \rightarrow v^j \]

\[
\begin{align*}
\Delta h_{jk}^{TT} & \approx \frac{2}{R} \mathcal{I}_{jk}^{TT} \\
\mathcal{I}_{jk}^{TT} & = \mu \left[ x_j x_k \right]^{TT} \\
\ddot{x}_j & = -\frac{M}{r^3} x_j \\
\mathcal{I}_{jk}^{TT} & = \mu \left[ x_j \ddot{x}_k + \ddot{x}_j x_k + 2 \dot{x}_j \dot{x}_k \right]^{TT} \\
& = 2\mu \left[ \dot{x}_j \dot{x}_k - \frac{M}{r^3} x_j x_k \right]^{TT}
\end{align*}
\]

\[ \Delta h_{jk}^{TT} = \frac{4\mu}{R} \Delta [v^j v^k]^{TT} \]
Understanding the memory: the linear memory effect

Hyperbolic orbit/two-body scattering

\[ h_{jk}^{TT} \approx \frac{2}{R} \mathcal{I}_{jk}^{TT} \]
\[ \mathcal{I}_{jk}^{TT} = \mu [x_j x_k]^{TT} \]

Hyperbolic orbit/two-body scattering

\[ \ddot{x}_j = -\frac{M}{r^3} x_j \]
\[ \ddot{\mathcal{I}}_{jk}^{TT} = \mu [x_j \ddot{x}_k + \ddot{x}_j x_k + 2\dot{x}_j \dot{x}_k]^{TT} \]
\[ = 2\mu \left[ \dot{x}_j \dot{x}_k - \frac{M}{r^3} x_j x_k \right]^{TT} \]

\[ \Delta h_{jk}^{TT} = \frac{4\mu}{R} \Delta \left[ v^j v^k \right]^{TT} \]
Understanding the memory: the linear memory effect

General formula for the memory jump in a system w/ N components [Braginsky & Thorne ‘87, Thorne ‘92]

\[ \Box \bar{h}_{ij} = -16\pi \sum_{A=1}^{N} T_{ij}^{pp,A} \]

\[ \Delta h_{ij} = \lim_{t \to +\infty} h_{ij}(t) - \lim_{t \to -\infty} h_{ij}(t) \]

\[ \Delta h_{ij}^{TT} = \Delta \sum_{A=1}^{N} \frac{4M_{A}}{R \sqrt{1 - v_{A}^2}} \left[ \frac{v_{A}^{j} v_{A}^{k}}{1 - N \cdot \mathbf{v}_{A}} \right]^{TT} \]
Understanding the memory: the nonlinear memory

[Mathematically, the nonlinear memory arises from the contribution of the gravitational-wave stress-energy to Einstein’s equations:]

\[ \square \bar{h}^{\alpha\beta} = -16\pi(-g)(T^{\alpha\beta} + t_{LL}^{\alpha\beta}) - \bar{h}^{\alpha\mu}_{,\nu} \bar{h}^{\beta\nu}_{,\mu} + \bar{h}^{\mu\nu} \bar{h}^{\alpha\beta}_{,\mu\nu} \]

Harmonic gauge EFE...

...has a nonlinear source from the GW stress-energy tensor.

Solve EFE:

\[ \bar{h}_{jk}(t, \mathbf{x}) = 4 \int \frac{(-g)[T_{jk}(t', \mathbf{x}') + t_{LL}^{jk}(t', \mathbf{x}')] + \ldots}{|\mathbf{x} - \mathbf{x}'|} \delta(t' - t - |\mathbf{x} - \mathbf{x}'|) d^4 x' \]

\[ \delta \bar{h}_{jk}^{TT} = \frac{4}{R} \int_{-\infty}^{T_R} dt' \left[ \int \frac{dE_{gw}}{dt' d\Omega'} \left( \frac{n_j' n_k'}{n_j n_k} \right) d\Omega' \right]^{TT} \]
Understanding the memory: the nonlinear memory

Mathematically, the nonlinear memory arises from the contribution of the gravitational-wave stress-energy to Einstein’s equations:

Nonlinear memory can be related to the “linear” memory if we interpret the component masses as the individual radiated gravitons (Thorne’92):

\[
\Delta h_{ij}^{TT} = \Delta \sum_{A=1}^{N} \frac{4M_A}{R\sqrt{1 - v_A^2}} \left[ \frac{v_A^j v_A^k}{1 - v_A \cdot N} \right]^{TT} \\
\Delta h_{ij}^{TT} = \Delta \sum_{A=1}^{N} \frac{4E_A}{R} \left[ \frac{n_A^j n_A^k}{1 - n_A \cdot N} \right]^{TT}
\]

\[
\frac{v_A^j}{c} \rightarrow n_A^j \\
\frac{M_A c^2}{\sqrt{1 - v_A^2}} \rightarrow E_A
\]

\[
\delta h_{jk}^{TT} = \frac{4}{R} \int_{-\infty}^{T_R} dt' \left[ \int \frac{dE_{gw}^{\prime}}{dt' d\Omega'} \frac{n_j' n_k'}{(1 - n' \cdot N) d\Omega'} \right]^{TT}
\]
Understanding the memory: the nonlinear memory

[Christodoulou ‘91; Blanchet & Damour ‘92]

Can also think of it as a nonlinear correction to the multipoles:

\[ T_{\alpha \beta}^{gw} \propto \frac{d E_{gw}}{d t d \Omega} \sim O(h^2) \]

\[ \mathcal{L}_{jk} \rightarrow \mathcal{L}_{jk} + U_{jk}^{gw} \]

- Memory piece scales like the radiated energy.

\[ h_{jk}^{TT} \approx \frac{2}{R} \mathcal{L}_{jk}^{TT} \]

- So the nonlinear memory is present in \textit{all} GW sources.

- The effect is hereditary (depends on entire past evolution).
Understanding the memory: the nonlinear memory: inspiralling binaries

Although it arises from a 2.5PN correction to the multipole moments, for inspiralling binaries the nonlinear affects the waveform at leading (Newtonian) order: 

$$h_+ = -2\frac{\mu}{R}v_{\text{orb}}^2 \left[ (1 + \cos^2 \Theta) \cos[2\varphi(t) - 2\Phi] + \frac{1}{36} \sin^2 \Theta (17 + \cos^2 \Theta) + O(v_{\text{orb}}^{1/2}) \right]$$

[Wiseman & Will ‘91]

Why?

$$\Delta h_{\text{mem}}^{jk} \sim \frac{\Delta E_{\text{GW}}}{R}$$

$$\Delta E_{\text{GW}} \sim \Delta E_{\text{binding}} \sim \frac{\mu M}{r} \sim \mu v_{\text{orb}}^2$$

$$h_{\text{oscil.}}^{ij} \propto \frac{1}{R} \ddot{T}_{ij} \sim \frac{\mu}{R} v_{\text{orb}}^2$$
Computing the nonlinear memory: previous/ongoing work (inspiral only)

✓ Wiseman & Will ‘91: 0PN memory waveform (circular, nonspinning).

✓ Thorne ‘92: analogy w/ linear memory; crude detectability estimates.

✓ Kennefick ‘94: repeats Wiseman-Will; crude detectability estimates.

✓ Wiseman & Will ‘91: nonlinear memory from high-velocity scattering (e>>1).

✓ Arun, Blanchet, Iyer, et al ‘04, ‘08: compute 3PN waveform; 0.5PN memory vanishes (circular, nonspinning).

✓ MF ’09a: 3PN memory waveform (circular, nonspinning).

✓ MF ’11: leading-order nonlinear memory for eccentric binaries (elliptical, hyperbolic, parabolic, radial; nonspinning); crude detectability estimates.

✓ Guo & MF (in prep): 1.5PN memory waveform (spinning binaries).
Computing the nonlinear memory: previous/ongoing work (inspiral only)

Calculation of the inspiral memory is important because it allow us to:

- obtain analytical understanding of how the memory behaves.
- complete our knowledge of PN waveforms consistently to a given order.
- provide accurate initial conditions for the memory in NR calculations.
Computing the nonlinear memory: previous/ongoing work (merging BHs)

For detectability purposes, we need to know the entire build-up and saturation value of the memory (need inspiral + merger/ringdown).

 ✓ MF ‘08, ‘09b: “minimal waveform model” & EOB calc; detectability estimates.
 ✓ MF (in prep): hybrid NR/PN calculation; improved detectability estimates.
 ✓ Pollney & Reisswig ‘11: extraction from full NR evolutions; aligned spins.
 ✓ Plans for memory search in LIGO.
Computing the nonlinear memory: outline of the calculation

1. Waveform can be expanded in spin-weighted spherical harmonic modes:

\[ h_+ - ih_\times = \sum_{l=2}^{\infty} \sum_{m=-l}^{l} h^{lm}(T_R, R) \cdot e^{-2Y^{lm}(\Theta, \Phi)} \]

2. The nonlinear memory modes are related to the GW energy flux:

\[ h_{lm}^{(\text{mem})} = \frac{16\pi}{R} \sqrt{(l - 2)! \frac{(l + 2)!}{(l - 2)!}} \int_{-\infty}^{T_R} dt \int d\Omega \frac{dE_{gw}}{dtd\Omega} (\Omega) Y^*_{lm}(\Omega) \]

3. The energy flux is related to the oscillating (non-memory) \( h_{lm} \) modes:

\[ \frac{dE_{gw}}{dtd\Omega} = \frac{R^2}{16\pi} \left\langle \dot{h}_+^2 + \dot{h}_\times^2 \right\rangle = \frac{R^2}{16\pi} \sum_{l',m',m''} \left\langle \dot{h}_{l'm'} \dot{h}_{l''m''}^* \right\rangle - 2Y^{l'm'} - 2Y^{l''m''} \]

4. Compute time-derivative of \( h_{lm}[v(t), L(t), S_1(t), S_2(t), e(t)] \), substitute equations of motion, plug in and integrate.
Computing the nonlinear memory: result: 3PN $h_{lm}$ modes and polarization

\[ h_{+,x} = \frac{2\eta Mx}{R} H_{+,x} + O \left( \frac{1}{R^2} \right), \text{ where } H_{+,x} = \sum_{n=0}^{\infty} x^{n/2} H_{+,x}^{(n/2)}. \]

\[ H_{+,x}^{(0,\text{mem})} = \alpha \frac{1}{96} s_\Theta^2 (17 + c_\Theta^2), \]

\[ H_{+,x}^{(0.5,\text{mem})} = 0, \]

\[ H_{+,x}^{(1,\text{mem})} = \alpha s_\Theta^2 \left[ \frac{-354241}{2064384} - \frac{62059}{1032192} c_\Theta^2 - \frac{4195}{688128} c_\Theta^4 + \left( \frac{15607}{73728} + \frac{9373}{8192} c_\Theta^2 \right) \eta^2 \right], \]

\[ H_{+,x}^{(1.5,\text{mem})} = 0, \]

\[ H_{+,x}^{(2,\text{mem})} = \alpha s_\Theta^2 \left[ \frac{-3968456339}{9364304} + \frac{3.704408173}{16852022912} c_\Theta^2 + \frac{122725}{18608} c_\Theta^4 + \frac{75601}{15925248} c_\Theta^6 + \left( -\frac{7169749}{18579456} \right) \eta^2 \right], \]

\[ h_{+,x}^{(3,\text{mem})} = -\alpha \frac{5\pi}{21504} (1 - 4\eta) s_\Theta^2 \left( 509 + 472 c_\Theta^2 + 39 c_\Theta^4 \right), \]

\[ H_{+,x}^{(3,\text{mem})} = \alpha s_\Theta^2 \left\{ \frac{69549016726181}{46146017820672} + \frac{6094001938489}{23073008910336} c_\Theta^2 - \frac{1416964616993}{15382005940224} c_\Theta^4 - \frac{2455732667}{78479622144} c_\Theta^6 - \frac{9979199}{2491416576} c_\Theta^8 + \frac{1355497856557}{149824733184} - \frac{3485\pi^2}{9216} - \left( -\frac{3769402979}{4682022912} - \frac{205\pi^2}{9216} \right) c_\Theta^2 + \frac{31566573919}{49941577728} c_\Theta^4 + \frac{788261497}{3567255552} c_\Theta^6 + \frac{302431}{9437184} c_\Theta^8 \right\} \eta^2 \left( \frac{5319395}{28311552} - \frac{24019355}{99090432} c_\Theta^2 - \frac{4438085}{3145728} c_\Theta^4 - \frac{3393935}{7077888} c_\Theta^6 - \frac{7835}{98304} c_\Theta^8 \right) \eta^2 \]

\[ + \left( \frac{1433545}{63700992} + \frac{752315}{15925248} c_\Theta^2 + \frac{129185}{2359296} c_\Theta^4 + \frac{389095}{1179648} c_\Theta^6 + \frac{9065}{131072} c_\Theta^8 \right) \eta^2 \right\}, \]
Computing the nonlinear memory: result: 3PN $h_{lm}$ modes and polarization

$$h_{+}^{\text{mem}} = \frac{2\eta M}{R} x H_+$$

$$x = (M\omega)^{2/3} \approx \frac{M}{r} \left[1 + O(c^{-2})\right].$$

- **OPN**
- **1PN**
- **2PN**
- **2.5PN**
- **3PN**
Computing the nonlinear memory: result: aligned spins, no precession

\[ h_{+}^{(\text{mem})} = \frac{2\eta M v^2}{R} \sum_{n=0}^{\infty} v^n H_{+}^{(n/2,\text{mem})}. \]

\[ H_{+}^{(0,\text{mem})} = \frac{1}{96}s_{\Theta}^2(17 + c_{\Theta}^2), \quad H_{+}^{(0.5,\text{mem})} = 0, \]

\[ H_{+}^{(1,\text{mem,nonspin})} = s_{\Theta}^2 \left[ -\frac{354241}{2064384} - \frac{62059}{1032192} c_{\Theta}^2 - \frac{4195}{688128} c_{\Theta}^4 + \left( \frac{15607}{73728} + \frac{9373}{36864} c_{\Theta}^2 + \frac{215}{8192} c_{\Theta}^4 \right) \eta \right] \]

\[ H_{+}^{(1.5,\text{mem,nonspin})} = 0, \quad H_{+}^{(1.5,\text{mem,spin})} = \frac{s_{\Theta}^2}{768} \sum_{i=1,2} \chi_i \kappa_i \left[ 369 \frac{m_i^2}{M^2} + 351 \eta + c_{\Theta}^2 \left( 23 \frac{m_i^2}{M^2} + 57 \eta \right) \right] \]

\[ \chi_i = |S_i|/m_i, \quad \kappa_i = |\hat{s}_i| \cdot \hat{L}, \quad s_{\Theta} = \sin \Theta, \quad c_{\Theta} = \cos \Theta, \quad M = m_1 + m_2, \quad \eta = \frac{m_1 m_2}{M^2} \]

\[ S_1 = \chi_1 m_1^2, \quad S_2 = \chi_2 m_2^2 \]

Spin contributes a \(~20\% - 30\%\) correction.

[Guo & MF (in prep)]
Computing the nonlinear memory:
result: precessing spins (small inclination approx.)

\[ h_{+}^{(\text{mem})} = \frac{2\eta M v^2}{R} \sum_{n=0}^{\infty} v^n H_{+}^{(n/2,\text{mem})}. \]

\[ H_{+}^{(0,\text{mem})} = \frac{1}{96} s_{\Theta}^2 (17 + c_{\Theta}^2), \quad H_{+}^{(0.5,\text{mem})} = 0, \]

\[ H_{+}^{(1,\text{mem,nonspin})} = s_{\Theta}^2 \left[ -\frac{354241}{2064384} - \frac{62059}{1032192} c_{\Theta}^2 - \frac{4195}{688128} c_{\Theta}^4 + \left( \frac{15607}{73728} + \frac{9373}{36864} c_{\Theta}^2 + \frac{215}{8192} c_{\Theta}^4 \right) \eta \right] \]

\[ H_{+}^{(1,\text{mem,spin})} = -s_{\Theta}^2 \sum_{i=1,2} \chi_i^2 (1 - \kappa_i^2) \frac{m_i^4}{M^4} \left( \frac{25 + 5c_{\Theta}^2}{192\eta^2} \right) \]

\[ H_{+}^{(1.5,\text{mem,nonspin})} = 0, \quad H_{+}^{(1.5,\text{mem,spin})} = \frac{s_{\Theta}^2}{768} \sum_{i=1,2} \chi_i \kappa_i \left[ 369 \frac{m_i^2}{M^2} + 351\eta + c_{\Theta}^2 \left( 23 \frac{m_i^2}{M^2} + 57\eta \right) \right], \]

\[ \chi_i = |S_i|/m_i \]
\[ \kappa_i = |\hat{s}_i| \cdot \hat{L} \]
\[ s_{\Theta} = \sin \Theta \]
\[ c_{\Theta} = \cos \Theta \]
\[ M = m_1 + m_2 \]
\[ \eta = \frac{m_1 m_2}{M^2} \]

\[ \iota \ll 1 \rightarrow |L| \gg |S| \]

[Guo & MF (in prep)]
Computing the nonlinear memory: result: arbitrary spins

- Need to solve spin equations numerically.

- Simple precession case: memory monotonically increases as before.

- Transitional precession (preliminary): mostly monotonic increase, except for a single oscillation during the “transition”

[ Apostolatos et al’94 ]

[ Guo & MF (in prep)]
Computing the nonlinear memory: eccentric binaries: motivation

- **Elliptical orbit (e<1) waveforms:**
  - 0PN (Peters-Mathews ’63)
  - 1PN (Junker-Schafer ’91)
  - 2PN (Gopakumar-Iyer ’02)
  - Nonlinear memory correction unknown

- **Hyperbolic/parabolic orbit (e≥1) waveforms:**
  - 0PN (Mike Turner ’77)
  - 1PN (Junker-Schafer ’91)
  - Nonlinear memory only computed for e>>1 case (Wiseman-Will ‘91)

- **Circularized binaries were eccentric (even hyperbolic) in the past:**
  - Since the nonlinear memory is hereditary, effects of the (past-growing) eccentricity could potentially have an effect on the value of the (late-time) memory.
Computing the nonlinear memory: eccentric binaries: outline of calculation

1. We want the leading-order nonlinear memory terms: these rely on only the leading-order mass-quadrupole moments for Keplerian orbits:

\[ h_{2m}^N \approx \frac{\ddot{I}_{2m}}{R \sqrt{2}} \]

2. These leading-order modes are substituted into an expression for the nonlinear memory modes:

\[ h_{lm}^{\text{mem}} = \frac{16\pi}{R} \sqrt{\frac{(l - 2)!}{(l + 2)!}} \int_{-\infty}^{T_R} dt \int d\Omega \frac{dE_{gw}}{dtd\Omega}(\Omega) Y_{lm}^*(\Omega) \]

\[ h_{20}^{\text{mem}} = \int_{-\infty}^{T_R} dt \frac{R}{42} \sqrt{\frac{15}{2\pi}} \left\langle 2|\dot{h}_{22}^N|^2 - |\dot{h}_{20}^N|^2 \right\rangle \]

3. To compute the time integral, use a Keplerian model for the orbit (including radiation-reaction in the elliptical case). Transform time-integral to an integral over the true-anomaly or the eccentricity.
Computing the nonlinear memory: result: eccentric binaries

linear memory case:

Hyperbolic orbits: \( \Delta h^{(\text{lin. mem})} \propto \eta \left( \frac{M}{R} \right) \left( \frac{M}{p} \right) \frac{(e^2 - 1)^{3/2}}{e^2} \)

Parabolic orbits: no linear memory

Elliptical orbits: no linear memory

Nonlinear memory case:

Hyperbolic/parabolic orbits: \( \Delta h^{(\text{mem})} \propto \eta^2 \left( \frac{M}{R} \right) \left( \frac{M}{p} \right)^{7/2} F(e) \)

Elliptical orbits:

\[
h_{20}^{(\text{mem})} = -\frac{2}{7} \sqrt{\frac{10\pi}{3 \eta M^2}} \frac{e_0^{12/19}}{R p_0} (304 + 121e_0^2)^{870/2299} \int_{e_0}^{e(t)} de \frac{1}{e^{31/19}} \frac{(192 + 580e^2 + 73e^4)}{(304 + 121e^2)^{3169/2299}}
\]

\[
\Delta h^{(\text{mem})} \propto \eta \left( \frac{M}{R} \right) \left( \frac{M}{p_0} \right) \left[ 1 - \left( \frac{e_0}{e(t)} \right)^{12/19} \right]
\]

[MF PRD’11]
Computing the nonlinear memory: result: eccentric binaries

Nonlinear memory case:

Hyperbolic/parabolic orbits: \( \Delta h^{(\text{mem})} \propto \eta^2 \left( \frac{M}{R} \right) \left( \frac{M}{p} \right)^{7/2} F(e) \)

Elliptical orbits:

\[ h_{20}^{(\text{mem})} = -\frac{2}{7} \sqrt{\frac{10\pi}{3}} \eta M^2 R p_0 e_0^{12/19} (304 + 121 e_0^2)^{870/2299} \int_{e_0}^{e(t)} de \frac{1}{e^{31/19}} \frac{(192 + 580 e^2 + 73 e^4)}{(304 + 121 e^2)^{3169/2299}} \]

\( \Delta h^{(\text{mem})} \propto \eta \left( \frac{M}{R} \right) \left( \frac{M}{p_0} \right) \left[ 1 - \left( \frac{e_0}{e(t)} \right)^{12/19} \right] \)

\[ 2.5\text{PN} \]

\[ \text{OPN} \]

[MF PRD’11]
Computing the nonlinear memory: result: sensitivity to early-time eccentricity

- Hereditary nature of memory implies dependence on past orbital evolution.
- BH binaries usually assumed to be circularized.
- Does the binary’s past-growing eccentricity affect memory calculations that assume quasi-circular orbits?

[MF PRD’11]
Summary of results in eccentric case:

• The nonlinear memory is a non-oscillatory component of the gravitational-wave signal; it is due to gravitational waves produced by gravitational waves.

• Previous calculations considered only quasi-circular binaries.

• Nonlinear memory waveforms have been computed for binaries with any eccentricity (elliptical, hyperbolic, parabolic, and radial orbits).

• In the hyperbolic/parabolic case, nonlinear memory enters at 2.5PN order.

• In the elliptic case, nonlinear memory enters at the same order as the Peters-Mathews waveforms (0PN order, as in the circular case).

• While the nonlinear memory depends on the past-history of the binary, the past-growing eccentricity is only a small correction to the memory for nearly-circularized binaries.
Calculating the nonlinear memory from BH mergers: limitations of numerical relativity

- No memory in (2,2) mode! Shows up in m=0 modes in quasi-circular case.

- In $\Psi_4$ memory is a 5PN order effect; in $h_+$ it is a 0PN effect.

(2,2), (4,4), (3,2), (4,2) modes are much larger than the memory modes (2,0), (4,0), etc..
Other problems with NR computations of the memory:

- Need to choose two integration constants to go from curvature to metric perturbation $\psi_{lm} = \hat{h}_{lm}$

  Choosing these incorrectly leads to “artificial” memory (Berti et al. ’07).

- Memory is sensitive to the past-history of the source; need large initial separation:
  
  &bullet; Consider leading-order (2,0) memory mode, with a finite separation $r_0$

  \[
  h_{20}^{NR}(T_R) = \frac{4}{7} \sqrt{\frac{5\pi}{6}} \frac{\eta M}{R} \left( \frac{M}{r(t)} - \frac{M}{r_0} \right)
  \]

  \[
  \frac{|\delta h_{20}^{NR}|}{h_{20}} \approx \frac{r(t)}{r_0}
  \]

- Errors from gauge effects and finite extraction radius can further contaminate NR waveforms and swamp a small memory signal.

[MF PRD’09]
Recall that the nonlinear memory contributes a piece to the $h_{lm}$ modes that depends on integrals over the gravitational-wave energy flux...

\[
h_{lm}^{(\text{mem})} = \frac{32\pi}{R\sqrt{2}} \sqrt{\frac{(l-2)!}{2(l+2)!}} \int_{-\infty}^{T_{R}} dt \int d\Omega \frac{dE_{gw}}{dtd\Omega}(\Omega)Y_{lm}^*(\Omega)
\]

...where this flux is itself a sum over the $h^{lm}$ modes:

\[
\frac{dE_{gw}}{dtd\Omega} = \frac{R^2}{16\pi} \langle \dot{h}_{+}^2 + \dot{h}_{\times}^2 \rangle = \frac{R^2}{16\pi} \sum_{l',l'',m',m''} \langle \dot{h}^{l'm'} \dot{h}^{*l''m''} \rangle -_2 Y^{l'm'} -_2 Y^{l''m''*}
\]

The nonlinear memory only appears in the $m=0$ modes, which can be expanded in terms of the $m\neq 0$ modes:

\[
h_{20}^{(\text{mem})} = \frac{R}{21} \sqrt{\frac{5}{2\pi}} \int_{-\infty}^{T_{R}} dt \left[ |\dot{h}_{22}|^2 + \frac{\sqrt{35}}{4} \left( \dot{h}_{22} \dot{h}_{32}^* + \dot{h}_{22}^* \dot{h}_{32} \right) + \text{higher order modes} \right]
\]

Plug in merger-ringdown description for $h_{22}$, etc...and then match to 3PN inspiral calculation of $h_{20}$. 

Calculating the nonlinear memory from BH mergers: an analytic hybrid scheme
Calculating the nonlinear memory from BH mergers: an analytic hybrid scheme: minimal waveform model

Only consider leading-order (2,2) contribution to the memory...

\[ h_{20}^{\text{mem}} = \frac{R}{21} \sqrt{\frac{5}{2\pi}} \int_{-\infty}^{T_R} dt |h_{22}|^2 \]

...and use a simple model for the inspiral and ringdown (2,2) modes...

\[ h_{22}^{\text{insp}} = -\frac{8}{R} \sqrt{\frac{\pi \eta m^2}{5}} \frac{e^{-2i\phi(t)}}{r} \quad h_{22}^{\text{ring}} = \frac{1}{R} \sum_{n=0}^{2} A_{22n} e^{-\sigma_{22n} t} \]

...matching the multipoles and 2 derivatives at the light-ring \((r/m \approx 3)\) to get \(A_{22n}\).

\(r(t)\) and \(\phi(t)\) evolve via the standard leading-order formulas:

\[ r(t) = r_m (1 - t/\tau_{rr})^{1/4} \quad \phi(t) = \sqrt{m/r^3} t + \phi_0 \]

Use NR results to determine the final mass and spin (as a function of \(\eta\)) that enter the quasi-normal mode (QNM) frequencies and damping times.

Get a fully analytic time-domain solution for the full memory:

\[ \hat{h}_{\text{MWM}}^{\text{mem}} = \frac{8\pi M}{r(T)} H(-T) + H(T) \left\{ \frac{8\pi M}{r_m} + \frac{1}{\eta M} \sum_{n,n' = 0}^{n_{\text{max}}} \frac{\sigma_{22n}\sigma_{22n'} A_{22n}A_{22n'}^{*}}{\sigma_{22n} + \sigma_{22n'}^{*}} \left[ 1 - e^{-(\sigma_{22n} + \sigma_{22n'}^{*})T} \right] \right\} \quad \text{[MF ApJL '09]} \]
Calculating the nonlinear memory from BH mergers: an analytic hybrid scheme: EOB model

Only consider leading-order (2,2) contribution to the memory...

\[ h_{20}^{(\text{mem})} = \frac{R}{21} \sqrt{\frac{5}{2\pi}} \int_{-\infty}^{T_R} dt \left| \dot{h}_{22} \right|^2 \]

Or use an EOB model calibrated to NR simulations:

\[ h_{22}^{\text{EOB,insp}} = -\frac{8M}{R} \sqrt{\frac{\pi}{5}} (r_\omega \Omega)^2 e^{2i\varphi} F_{22} F_{22}^{\text{NQC}} \]

Solve EOB Hamilton’s equation for particle motion, and match inspiral modes to 5 QNMs.

Calculating the nonlinear memory from BH mergers: an analytic hybrid scheme: direct NR-hybrid

Only consider leading-order (2,2) contribution to the memory...

\[
h_{20}^{(\text{mem})} = \frac{R}{21} \sqrt{\frac{5}{2\pi}} \int_{-\infty}^{T_R} dt |\dot{h}_{22}|^2
\]

Or directly use the numerical NR (2,2) mode [Caltech/Cornell/CITA]:

~27% smaller than previous EOB calculation.

smooth matching to 3PN order inspiral allows evaluation to arbitrarily early times.
Memory during the merger/ringdown: NR results

Pollney & Reissweg [arXiv:1004.4209] extract memory directly from NR simulation:

- causally-disconnected outer boundary
- Cauchy characteristic extraction to compute GWs at Scri+
- equal-mass, equal spins aligned or anti-aligned.
- excitation of (2,0) QNM.
- accurate 3PN inspiral memory crucial to getting correct magnitude.
Detectability of the nonlinear memory: interferometers
Detectability of the nonlinear memory: LISA
Detectability of the nonlinear memory: pulsar timing

- Memory burst detectable out to $z \sim 0.1$ for $M=10^8 \, M_\odot$ (observable universe for $M=10^{10} \, M_\odot$).

- But rates are low: $0.1 - 0.01$ detections in 10 yrs w/ near-term Pulsar-Timing-Arrays.

Seto, MNRAS ’09
van Haasteren & Levin, MNRAS ‘10
Additional DC contributions to the waveform:

• Arun et al ’04 found a **nonlinear, non-hereditary** DC contribution to the “×” polarization at 2.5PN order; it originates from nonlinear corrections to the radiative current octupole moment $V^{3m}$:

$$H_{x}^{(2.5, \text{mem})} = -\alpha \frac{6}{5} s_\Theta^2 c_\Theta \eta$$

• In addition, there are **linear, zero-frequency** terms that arise from the $m=0$ pieces of the source mass and current moments. For example:

$$I_{lm} \propto \eta M r^l(t) e^{-im\varphi(t)} [1 + O(2)]$$

$$I_{2\pm 2}^{(2)} \propto \eta M x e^{\pm 2i\varphi(t)} [1 + O(2)]$$

$$I_{20}^{(2)} \propto \eta^3 M x^6 [1 + O(2)]$$

...which leads to a 5PN, non-oscillatory correction to the waveform.

• Not clear if these effects produce a memory that continues to build up during the merger/ringdown.
Summary:

1. The linear memory arises from systems w/ unbound masses (hyperbolic orbits, explosions).

2. The nonlinear memory arises from the GWs produced by GWs, and is a generic feature of all GW sources.

3. For quasi-circular and elliptical inspiraling binaries, the nonlinear memory causes a slowly-growing, non-oscillatory amplitude correction to the waveform at leading-(Newtonian)-order.

4. Modeling the memory in BH mergers is difficult for most NR codes, but can be accomplished using quasi-analytic techniques.

5. The memory is detectable by LISA for large SNR mergers. Detection prospects are poor for the upcoming generation of ground-based detectors, but not substantially worse than other classes of sources that we routinely try to detect.

6. Observation of the nonlinear memory provides a means to confirm an interesting strong-field prediction of GR.