Due: March 22nd, 2024,10pm
Problems Due: 1,4,5

1. B.Math(hons.) first year class are trying to determine the ideal amount of coffee to be sold at the student canteen. They randomly sample 8 students in the canteen and ask them how much ml of coffee they would like to drink. The data is shown below.

   $$300, 500, 300, 700, 300, 500, 300, 700.$$

   Assuming normality and variance to be 100, construct a 95% confidence interval for the true population mean of the preferred drinking amount using the above data.

2. Let $\theta > 0$ and $X_1, X_2, X_3 \ldots, X_n$ be from the Uniform$(0, \theta)$ distribution.

   (a) Find the maximum likelihood estimate $\hat{\theta}$ of θ.

   (b) Consider the piovtal quantity

   $$T(X_1, X_2, \ldots, X_n, \theta) = \frac{\hat{\theta}}{\theta}.$$

   Find the distribution of T.

   (c) Using T, find a 95% confidence interval for θ.

3. Suppose we collect a simple random sample of size 19 and find sample mean is 12 and sample standard deviation is 6.3. Under assumption of Normality can you find a 95% confidence interval for the true mean ?

4. Suppose $X_1, X_2, X_3, \ldots X_n$ be i.i.d Bernoulli(p). Using the Central Limit Theorem can you find a 95% confidence interval for p ?

5. Let $\sigma > 0$ and $X_1, X_2, X_3, \ldots X_n$ be an i.i.d. sample from Normal$(0, \sigma^2)$ distribution.

   (a) Find the distribution of

   $$T(X_1, X_2, \ldots, X_n, \sigma^2) = \sum_{i=1}^{n} \frac{X_i^2}{\sigma^2},$$

   (b) Using `qchisq()` in R find 95% -confidence intervals with equal tail exclusion probabilites.

   (c) Using `qchisq()` in R find 95% -confidence intervals with one-sided intervals.

6. Let $\theta > 0$ be unknown and $X_1, X_2, \ldots, X_n$ be an i.i.d. sample from a population with Cauchy$(\theta, 1)$ distribution.

   (a) Find the likelihood function $L(\theta, X_1, X_2, \ldots, X_n)$.

   (b) Let $\ell(\theta\,; X_1, X_2, \ldots, X_n) = \log(L(\theta\,; X_1, X_2, \ldots, X_n))$. Decide that one cannot explicitly compute the critical points on $\ell(\cdot)$.

   (c) Can you write an R-code to numerically find the maximum likelihood estimate for θ.

Book-Keeping Exercises[1]

1. Let $0 < p < 1$ and $X_1, X_2, \ldots X_n$ be an i.i.d. sample from a population with Geometric(p) distribution.
   (a) Find the likelihood function $L(p\,; X_1, X_2, \ldots, X_n)$.
   (b) Let $l(p\,; X_1, X_2, \ldots, X_n) = \log(L(p\,; X_1, X_2, \ldots, X_n))$. Find the value of p for which $l(p\,; X_1, X_2, \ldots, X_n)$ is maximized.
   (c) Prove that the maximum likelihood estimate for p is $1/\bar{X}$.
   (d) Can you find estimator for m and p in the event they are both unknown ?

2. Let m be a known integer and $p \in (0, 1)$ be an unknown parameter. Let $X_1, X_2, \ldots, X_n$ be an i.i.d. sample from a population with Binomial(m, p) distribution.
   (a) Find the likelihood function $L(p, X_1, X_2, \ldots, X_n)$.
   (b) Let $l(p\,; X_1, X_2, \ldots, X_n) = \log(L(p\,; X_1, X_2, \ldots, X_n))$. Find the value of p that maximizes $l(p\,; X_1, X_2, \ldots, X_n)$.
   (c) Prove that the maximum likelihood estimate for p is $\bar{X}/m$.

3. Let Z_1 and Z_2 be independent standard Normal random variables. Suppose,
$$\begin{aligned} U &= \sigma_1 Z_1 + \mu_1, \\ V &= \sigma_2(\rho Z_1 + \sqrt{1-\rho^2} Z_2) + \mu_2, \end{aligned}$$
   where $\sigma_1, \sigma_2 > 0$, $\mu_1, \mu_2 \in \mathbb{R}$ and $-1 < \rho < 1$. Find the joint density of U, V.

4. Let $a, b : \mathbb{N} \to \mathbb{R}_+$ be two sequences
   - $a_n = O(b_n)$ if there exists $N_0 \in \mathbb{N}$ and $c > 0$ such that $a_n \leq c b_n$ for all $n \geq N_0$
   - $a_n = o(b_n)$ if for every $\epsilon > 0$ there exists N_0 such that $a_n \leq \epsilon b_n$ for all $n \geq N_0$

   Consider the following sets of sequences:
   (a) $a_n = n^3 + 5n^2 + 15$ and $b_n = n^3 + 7n + 8$
   (b) $a_n = n^3 + 5n^2 + 15$ and $b_n = n^3 + 700000n + 1000$
   (c) $a_n = n^3 + 5n^2 + 15$ and $b_n = 0.0005n^4 + 7n^2 + 8$
   (d) $a_n = n^3 + 5n^2 + 15$ and $b_n = 2^n$
   (e) $a_n = n^5$ and $b_n = n^{10}$
   (f) $a_n = nr^n$, for $r \in (0, 1)$ and $b_n = \frac{1}{n^4}$
   (g) $a_n = \frac{2^n}{n!}$ and $b_n = n^{-0.9n}$
      i. Write an R code to obtain the plots of a_n/b_n.
      ii. From the plots can you guess for each of (a-g), if $a_n = O(b_n)$ or $a_n = o(b_n)$.
      iii. Decide (with proof) whether $a_n = O(b_n)$ or $a_n = o(b_n)$ for sequences in (a-g).

5. Let $X_1, X_2, X_3, \ldots, X_n$ be i.i.d. random variables with finite expectation μ, finite variance σ^2, and finite $\gamma = E[X_1 - \mu]^4$. Compute $Var(S^2)$ in terms of μ, σ^2, and γ and show that $Var(S^2) \to 0$ as $n \to \infty$.

[1] Questions 1 and 2 are from material within the course syllabus and you may be tested on these.