Due: March 7th, 2024,10pm
Problems Due: 2,3

1. (*t_n-distribution*) A random variable T is said to follow the students t distrbution with n degrees of freedom (denoted by t_n) if it has a probability density function given by

$$f(t) = \frac{\Gamma(\frac{n+1}{2})}{\sqrt{n\pi}\Gamma(\frac{n}{2})}\left(1+\frac{t^2}{n}\right)^{-\frac{n+1}{2}},$$

for $t \in \mathbb{R}$.

   (a) Take 100 equally spaced points in $[-3,3]$, using the `dt` and `dnorm` functions on *one graph* plot the probability density functions of $t_1, t_3 t_8, t_{30}$ and Normal$(0,1)$.

   (b) Using the `rt` and `rnorm` functions generate 100 samples from $t_1, t_3 t_8, t_{30}$ and Normal$(0,1)$ and compare the box plots of the samples from $t_1, t_3 t_8, t_{30}$ with samples from Normal$(0,1)$.

2. Suppose that U and V are independent Uniform$(0,1)$, and interpret them as coordinates of a point in $\mathbb{R}^2$. We wish to estimate the probability that the point (U,V) is inside the unit circle.

   (a) Let $Z = \sqrt{U^2+V^2}$. Find[1] the $P(Z<1)$.

   (b) Explain the computation performed in the below `R-code` and (justify) compare the output with answer obtained in part (a).

```
> replicate(10, {
+   u <- runif(10000)
+ v <- runif(10000)
+ sum(u^2+v^2 < 1) / 10000
+ })
```

   (c) Can we modify the above code to estimate π ?

3. *The R functions `histogram()` and `qqmath()` in the `lattice` package can be used to generate histograms and Q-Q plots respectively.*

   (a) Execute and Explain what the below `R-code` is doing:

```
> x.means <- replicate(1000, mean(rnorm(50)))
> library(package = "lattice")
> histogram(~ x.means, nint = 15)
> qqmath(~ x.means, grid = TRUE)
```

   What can you infer from its output ?

   (b) Using `lattice` (or otherwise) replicate 100 trials of computing $\bar{X}$ = `sample mean` and S^2=`sample variance` of 100 samples from Normal(0,1). Compute $10*\bar{X}/S$, the t-statistic for each replication. Compare its histogram with the p.d.f of the t_{99} distribution.

4. Let P be the empirical distribution defined by sample observations $X_1, X_2, X_3, \ldots, X_n$. Let Y be a random variable with distribution P.

   (a) Show that $E[Y] = \bar{X}$.

   (b) Show that $Var[Y] = \frac{n-1}{n}S^2$.

[1] you may use the fact the integral of constant function 1 over a region in $\mathbb{R}^2$ is the area of the region.

Book Keeping Exercises

1. Let $n \geq 1$ and let $X_1, X_2, \ldots, X_n$ be a i.i.d random sample from uniform $(0, 1)$. Let the X's be arranged in increasing order of magnitude denoted by

$$X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}.$$

These ordered values are called the order statistics of the sample $X_1, X_2, \ldots, X_n$. For, $1 \leq r \leq n$, $X_{(r)}$ is called the r-th order statistic. Using the method used in [AST16, Example 8.1.4],

   (a) find the probability density function of, $X_{(r)}$, the r-th order statistic for $1 \leq r \leq n$ and

   (b) find the probability density function of the range $R = X_{(n)} - X_{(1)}$.

2. Suppose X is a Normal $(0, 1)$ random variable and Y is an independent χ^2_m random variable. Let

$$T = \frac{X}{\sqrt{\frac{Y}{m}}}.$$

Then T is said to have the t_m distribution.

   (a) Using the method in [AST16, Example 8.1.5] show that the $U = X^2$ is a χ^2_1 random variable, i.e. it has a probability density function given by

$$f_U(u) = \begin{cases} \frac{1}{\sqrt{2\pi}} e^{-\frac{u}{2}} u^{-\frac{1}{2}} & \text{for } u \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

   (b) Using the method in [AST16, Example 5.5.10] show that the $F = \frac{\frac{U}{1}}{\frac{Y}{m}}$ is a $F(1, m)$ random variable, i.e. it has a probability density function given by

$$f_F(w) = \begin{cases} \frac{\Gamma(\frac{m+1}{2})}{\Gamma(\frac{m}{2})\sqrt{m\pi}} \cdot \frac{1}{\sqrt{w}(1+\frac{w}{m})^{\frac{m+1}{2}}} & \text{for } w \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

   (c) Show that T has probability density function given by

$$f_T(t) = \frac{\Gamma(\frac{m+1}{2})}{\Gamma(\frac{m}{2})\sqrt{m\pi}} \left(1 + \frac{t^2}{m}\right)^{-\frac{m+1}{2}} \qquad \text{for } t \in \mathbb{R}$$

References

[AST16] Siva Athreya, Deepayan Sarkar, and Steve Tanner. Probability and statistics with examples using r. 2016. Unfinished Book, Last Compilation April 25th 2016, available at `http://www.isibang.ac.in/~athreya/psweur/index.html`.