Estimation:

Example: Given a coin.
Toss it 100 times
Find that 67 heads appeared

Q:- What is the probability of heads of the coin?

"Best" - Guess: 0.67

- How good is the estimate?
- Can we specify an interval around 0.67 say (0.66, 0.68) where p belongs with "high" probability?
- $p = \frac{1}{2}$ (Claim made by manufacturer)

Can we device a test to reject $p = \frac{1}{2}$ from data 67 heads in 100 tosses?

Two Methods to Estimate a parameter

Assumptions

(a) $X_1, X_2, \ldots, X_n$ are i.i.d. X and



(b) $f(\cdot) \equiv f(\cdot \mid p)$ will depend on parameters $p \equiv (p_1, p_2, \ldots, p_d) \in \mathbb{R}^d$, $d \geq 1$

Example:

$X \sim$ Bernoulli (p)

$$f(x \mid p) = \begin{cases} p^x (1-p)^{1-x} & x \in \{0,1\} \\ 0 & \text{otherwise} \end{cases}$$

$X \sim$ Normal (μ, σ^2)

$$f(x \mid \mu, \sigma) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}} \qquad x \in \mathbb{R}$$

(1) Method of moments

$d \geq 1$

$X \sim f(\cdot \mid p)$ $\quad p = (p_1, p_2, \ldots, p_d) \in \mathbb{R}^d$

$X_1, \ldots, X_n$ for $n \geq 1$ X_i all iid X.

- $k \geq 1$, Empirical moments

$$m_k(X_1, \ldots, X_n) = \frac{1}{n} \sum_{i=1}^{n} X_i^k$$

- $k \geq 1$, moments

$$\mu_k = E[X^k] = \begin{cases} \sum_{x \in \text{Range}(X)} x^k f(x \mid p) \\ \int x^k f(x \mid p)\, dx \end{cases}$$

$\mu_k = \mu_k(p_1, \ldots, p_d)$

The method of moments estimator for β is obtained by solving

$$\mu_k(\beta_1, \ldots, \beta_d) = m_k(X_1, \ldots, X_n) \quad k = 1, \ldots, d \quad \} (*)$$

Remarks:

(1) $(*)$ provides d - equations in d - unknowns given a sample $X_1, \ldots, X_n$

(2) no guarantee of solution or unique or compute it

Denote solutions: $\hat{\beta}_1, \hat{\beta}_2 \ldots, \hat{\beta}_d$ $\equiv$ they are realised as functions of $X_1, \ldots, X_n$.

Example: $X_i \overset{d}{=} \text{Binomial}(N, p)$

$i = 1, \ldots, 10$

and values

8, 7, 6, 11, 8, 5, 3, 7, 6, 9

($\uparrow X_1$ ... $\uparrow X_{10}$)

$$m_1(X_1, \ldots, X_{10}) = \frac{1}{10} \sum_{i=1}^{10} X_i = 7$$

$$m_2(X_1, \ldots, X_{10}) = \frac{1}{10} \sum_{i=1}^{10} X_i^2 = 53.4$$

$$\mu_1(N, p) = E[X_1] = Np$$

$$\mu_2(N, p) = E[X_1^2] = Np(1-p) + (Np)^2$$

$$(\text{Var}[X_1] = EX_1^2 - (EX_1)^2)$$

Set:

$$Np = 7$$

$$Np(1-p) + (Np)^2 = 53.4$$

$\hat{N} = 18.8$ and $\hat{p} = 0.371$

$\hat{N} \simeq 19$ are Method of moment Estimators

II Maximum likelihood Estimator

Given

$X_1, \dots, X_n$ i.i.d. $X \equiv f(\cdot \mid p)$

$p \in (p_1, p_2, \dots, p_d) \in \mathcal{P}_d \subseteq \mathbb{R}^d$

independent and identically distributed.

Define likelihood $L: \mathbb{R}^d \to \mathbb{R}$

$$L(p \mid x_1, \dots, x_n) = \text{"Probability of observing } x_1, \dots, x_n\text{"}$$

$$= \prod_{i=1}^{n} f(x_i \mid p)$$

Maximum likelihood estimator (MLE) $\hat{p}$ is

$$\arg\max_{p \in \mathcal{P}^d} L(p \mid x_1, \dots, x_n)$$

[E.g. $L(p) = -p^2 + 1 \Rightarrow \hat{p} = 0$]

Remarks:

- $\hat{p}$ – may not exist value
- $\hat{p} = \hat{p}(X_1, \ldots, X_n)$

Example:

$X \sim \text{Normal}(p, 1)$

$X_1, X_2, \ldots, X_n$ are i.i.d X.

$$L[p \mid x_1, \ldots x_n] = \prod_{i=1}^{n} \frac{e^{-\frac{(x_i - p)^2}{2}}}{\sqrt{2\pi}}$$

$p \in \mathbb{R}$

$$= \frac{1}{(\sqrt{2\pi})^n} e^{-\sum_{i=1}^{n} \frac{(x_i - p)^2}{2}}$$

Ex:-

- $\underset{p}{\operatorname{argmax}}\, L(p \mid x_1, \dots x_n) =$

$$\underset{p}{\operatorname{argmin}} \sum_{i=1}^{n} (x_i - p)^2$$

- $$\sum_{i=1}^{n} (x_i - p)^2 = \sum_{i=1}^{n} (x_i - \bar{X})^2 + \sum_{i=1}^{n} (\bar{X} - p)^2$$

$$\bar{X} = \frac{1}{n} \sum_{i=1}^{n} x_i$$

$$\Rightarrow \quad \hat{p} = \underset{p \in \mathbb{R}}{\operatorname{argmin}} \sum_{i=1}^{n} (x_i - p)^2 = \bar{X}$$

is the MLE of p.