Recall:-

Reflection Principle

$a, c \in \mathbb{N} \cup \{0\}$

$$\mathbb{P}(S_n = a+c) = \mathbb{P}(\sigma_a \le n, S_n = a-c)$$

(Escape from Origin)

$$\mathbb{P}(\sigma_0 > 2n) = \mathbb{P}(S_{2n} = 0)$$

Ruin Problem:

Two players. Each with capital $\begin{cases} a - \text{I} \\ b - \text{II} \end{cases}$

Interpret

$S_n \equiv$ gain of player 1

- $\gamma^1 = \lim_{N\to\infty} \mathbb{P}_N(\text{Ruin of Player 1})$ } Existing
- $\gamma^2 = \lim_{N\to\infty} \mathbb{P}_N(\text{Ruin of Player 2})$

$$\gamma^1 = \frac{b}{a+b} \qquad \gamma^2 = \frac{a}{a+b} \quad !$$

Game System: $V_1, \dots, V_N : \Omega_N \to \mathbb{R}$ are random variables. such that

$$\{V_k = c\} \in \mathcal{A}_{k-1} \quad c \in \mathbb{R}, \ k = 1, 2, \dots N$$

$$S_n^V = \sum_{k=1}^{N} V_k X_k$$

Theorem 2: (impossibility of profitable games)
For any $V_1, V_2, \dots, V_N$ game system
then $\mathbb{E}[S_N^V] = 0$

Theorem 1:

- $V_k = 1_{\{T \ge k\}}$ $\qquad \mathbb{E}[S_T] = 0$

$V_k = S_{k-1} 1_{T \ge k}$ $\qquad \mathrm{Var}[S_T] = \mathbb{E}[T]$

1.5 Arc Sine law - last visit to origin in 2N time

$$L = \max\{0 \le n \le 2N \mid S_n = 0\}$$

(not a stopping time)

"Order of L" : time after which one player has gain in capital

$$L \approx 2N$$

Theorem 5: $n \in \mathbb{N}$ upto $n \le N$

$$\mathbb{P}(L = 2n) = \frac{1}{2^{2N}} \binom{2n}{n} \binom{2N-2n}{N-n}$$

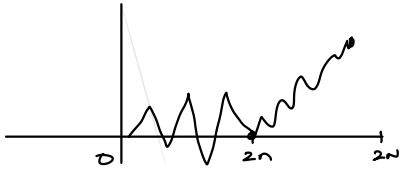
Proof: (next class)

(next class) $\mathbb{P}(L=2n)$ (stirling formula) $\simeq \frac{1}{N} f\left(\frac{n}{N}\right)$

$n! \sim n^n e^{-n}$

Proof :-

# of paths of length $2N$ with $L = 2n$



$= \left(\#\text{ of paths of length } 2n \text{ with } S_{2n} = 0\right) \times \left(\#\text{ of paths of length } 2N-2n \text{ with } \sigma_0 > 2N-2n\right)$

1st $2n$ tosses

next $2N-2n$ tosses

$$\therefore \quad \mathbb{P}(L = 2n) = \mathbb{P}(S_{2n} = 0)\ \mathbb{P}(\sigma_0 > 2N-2n)$$

$$\left(\begin{matrix}\text{lemma}\\ 4\end{matrix}\right) \leftarrow = \mathbb{P}(S_{2n}=0)\ \mathbb{P}(S_{2N-2n}=0)$$

$$= \binom{2n}{n} \frac{1}{2^{2n}} \binom{2N-2n}{N-n} \frac{1}{2^{2N-2n}}$$

$$= \underbrace{\frac{1}{2^{2N}} \binom{2n}{n} \binom{2N-2n}{N-n}}_{\text{Discrete arc Sine law}} \quad \square$$

Sketch (Ex: Make it precise)

$$\mathbb{P}(L=2n) = \leftarrow \boxed{\text{Stirling's approximation } n! \sim n^n e^{-n}}$$

$$\overset{\text{Ex}}{\simeq} \frac{1}{\pi} \frac{1}{\sqrt{n(N-n)}}$$

$$= \frac{1}{\pi} \cdot \frac{1}{2} \frac{1}{\sqrt{\frac{n}{2N}\left(1-\frac{n}{2N}\right)}}$$

$f: [0,1) \to \mathbb{R}$

$f(x) = \dfrac{1}{\pi\sqrt{x(1-x)}}$

will involve constants

$$\boxed{\mathbb{P}(L=2n) \simeq \frac{1}{2} f\left(\frac{n}{2N}\right)}$$

Intuition : $\{S_n\}_{n \geq 1}$ interpretation vis a vis game

L – time at which particular takes the lead for ever.

$E[S_{2N}] = 0 \Rightarrow L \simeq 2N$

Distribution function of scaled random variable

$$\mathbb{P}\left(\frac{L}{2N} \leq x\right) = \sum_{n \leq \lfloor 2Nx \rfloor} \mathbb{P}(L = 2Nn)$$

$$\sim \sum_{\substack{n \geq 1 \\ \frac{n}{2N} \leq x}} \frac{1}{2N} f\left(\frac{n}{2N}\right)$$

Riemann approximation $\left(\xrightarrow[N \to \infty]{}\right)$

$$\simeq \int_0^x f(y)\, dy$$

$$= \int_0^x \frac{1}{\pi \sqrt{y(1-y)}}\, dy$$

$$= \frac{2}{\pi} \arcsin\left(\sqrt{x}\right)$$

1.6 Infinite length random walks

2.1 Higher dimension finite length walks

$$\mathbb{Z}^d = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{bmatrix} \;\middle|\; x_i \in \mathbb{Z} \quad 1 \le i \le d \right\}$$

Fix $N \ge 1$, $\qquad |x| = \sqrt{\sum_{i=1}^{d} x_i^2}$

$$\Omega_N = \{ \omega = (\omega_1, \dots, \omega_N) : \omega_k \in \mathbb{Z}^d, \ |\omega_k| = 1, \ 1 \le k \le N \}$$

$$X_k : \Omega_N \to \mathbb{Z}^d \qquad X_k(\omega) = \omega_k$$

$$S_n = \sum_{i=1}^{n} X_i(\omega) \quad ; \ 1 \le n \le N$$

$$S_n = \begin{bmatrix} S_n^{(1)} \\ \vdots \\ S_n^{(d)} \end{bmatrix} \qquad 0 \le n \le N, \quad S_n^{(j)} \in \mathbb{Z}, \ 1 \le j \le d$$

$$\mathbb{P}_N(A) = \frac{|A|}{(2d)^N} \qquad A \subseteq \Omega_N$$

Observe that our construction of finite length walks of length N had the following property:

$0 < N < M$

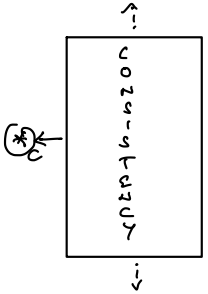
$\mathbb{P}_N(\cdot)$ $\qquad$ $\mathbb{P}_M(\cdot)$

$$\omega = (\underbrace{\omega_1, \dots, \omega_N}_{\vec{\omega} \in \Omega_N}, \omega_{N+1}, \dots, \omega_M) \in \Omega_M$$

$$\Pi_N : \Omega_M \to \Omega_N, \qquad \omega \to \vec{\omega}$$

Fix $\vec{\omega} \in \Omega_N$:

$$\mathbb{P}_M\left(\{w \in \Omega_M : \Pi_N(w) = \vec{\omega}\}\right)$$



$$= \frac{|\{w \in \Omega_M : \Pi_N(w) = \vec{\omega}\}|}{(2d)^M} = \frac{(2d)^{M-N}}{(2d)^M}$$

$$= (2d)^{-N}$$

$$\mathbb{P}_N(\{\vec{\omega}\}) = \frac{1}{(2d)^N}$$

Construction for $N \equiv \infty$ case

$$\Omega_\infty = \left\{ (\omega_m)_{m=1}^{\infty} \;\middle|\; \omega_m \in \mathbb{Z}^d,\ |\omega_m| = 1,\ 1 \le m \right\}$$

$$X_k : \Omega_\infty \to \mathbb{Z}^d \qquad X_k(\omega) = \omega_k \qquad 1 \le k$$

$$S_n = \sum_{k=1}^{n} X_k \qquad n \ge 1$$

Define: $\mathbb{P}$ on $(\Omega_\infty, \mathcal{F}_\infty)$ $\rightsquigarrow$ { exist

$\forall N \ge 1$, $\vec{\omega} \in \Omega_N$

$$\mathbb{P}\left(\{\omega \in \Omega_\infty \mid \Pi_N(\omega) = \vec{\omega}\}\right) := \frac{1}{(2d)^N} = \mathbb{P}_N(\{\vec{\omega}\})$$

Kolmogorov Consistency Theorem:

If $(\Omega_N, \mathcal{P}(\Omega_N), \mathbb{P}_N)$ satisfying (*)C Then

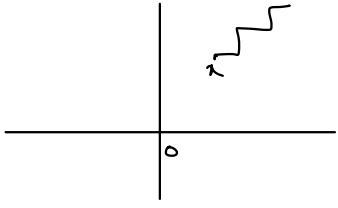
$\mathbb{P}$ on $(\Omega_\infty, \mathcal{F}_\infty)$ such $\mathbb{P}|_{\Omega_N} = \mathbb{P}_N$

Notation / Convention:

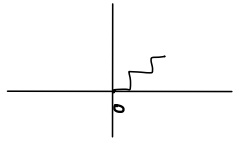
- $(\Omega_\infty, \mathbb{P})$ and $\{S_n\}_{n\geq 1}$ as simple random walk starting at 0.

- $\{S_n\}_{n\geq 1}$ to start at x, define another Probability $\mathbb{P}_x(\cdot)$

$$\mathbb{P}_x(S_1 = s_1, \ldots, S_n = s_n) = \mathbb{P}(S_1 + x = s_1, \ldots, S_n + x = s_n)$$



Reference frame



$$\mathbb{E}_x[f(S_1, \ldots, S_n)] = \mathbb{E}[f(S_1 + x, \ldots, S_n + x)]$$

Generalisations:

- Each step is of size 1 / length 1
- Probability of each step = same

Independence

- (Both length & equal probabilities can be tweaked)

Strong law of large numbers: $\{S_n\}_{n\geq 1}$ is a simple random walk on $\mathbb{Z}^d$.

$$A = \{\omega \in \Omega_0 \mid \frac{S_n}{n} \longrightarrow 0 \text{ as } n \to \infty\}$$

then $\mathbb{P}(A) = 1$.

Typical position of the walk

$d=1$ $\{S_n\}_{n\geq 1}$ is simple random walk on $\mathbb{Z}$.

Central limit theorem:

$$\mathbb{P}\left(\frac{S_n}{\sqrt{n}} \leq x\right) \longrightarrow \int_{-\infty}^{x} \frac{e^{-y^2/2}}{\sqrt{2\pi}}\, dy$$

as $n \to \infty$