Due: Wednesday, October 8th, 2025
Time and Problems due: will be mentioned in class

1. Let X, Y, Z be discrete random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Suppose that $\mathbb{P}(Y = y) > 0$ and $\mathbb{P}(Z = z) > 0$.
   (a) Show that $\mathbb{E}[XY \mid Y = y] = y\mathbb{E}[X \mid Y = y]$
   (b) If X is independent of Y then $\mathbb{E}[X \mid Y = y] = \mathbb{E}[X]$ and hence $\mathbb{E}[X \mid Y] = \mathbb{E}[X]$
2. Let X, Y, Z be discrete random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Suppose that $\mathbb{P}(Y = y) > 0$ and $\mathbb{P}(Z = z) > 0$.
   (a) Show that $\mathbb{E}[X + Y \mid Y = y] = y + \mathbb{E}[X \mid Y = y]$
   (b) Show that $\mathbb{E}[Y \mid Y = y, Z = z] = y$
3. Let $X \sim$ Uniform $\{1, 2, \ldots, n\}$ be independent of $Y \sim$ Uniform $\{1, 2, \ldots, n\}$. Let $Z = \max(X, Y)$ and $W = \min(X, Y)$.
   (a) Find the joint distribution of (Z, W).
   (b) Find $E[Z \mid W]$.
4. Let $\Omega = \{-1, +1\}^{\mathbb{N}}$ equipped with the probability denoted by $\mathbb{P}$, such that

$$\mathbb{P}(\{\omega \in \Omega : \pi_N(\omega) = \tilde{\omega}) = \frac{1}{2^N} = \mathbb{P}_N(\{\tilde{\omega}\}),$$

   where $N \in \mathbb{N}$, $\tilde{\omega} \in \Omega_N = \{-1, 1\}^N$, equipped with the uniform distribution, denoted by $\mathbb{P}_N$ and $\pi_N : \Omega \to \Omega_N$ be the cannonical projection.

   Consider for $k \geq 1$, $X_k : \Omega \to \{-1, 1\}$ be given by $X_k(\omega) = \omega_k$ and for $1 \leq n$, let $S_n : \Omega \to \mathbb{Z}$ be given by $S_n(\omega) = \sum_{k=1}^n X_k(\omega)$ and $S_0 = 0$.

   Definition: *Let $\mathcal{A}_n$ be the events that are observable by time n. We shall say a sequence of random variables, $\{H_n\}$ is a martingale w.r.t. the filtration $\mathcal{A}_n$ if*

$$\mathbb{E}[\mid H_n \mid] < \infty \quad \textit{and} \quad \mathbb{E}[H_n \mid S_{n-1}, \ldots, S_1] = H_{n-1}$$

   (a) Show that $\xi_n = S_n^2 - n$ is a martingale w.r.t the filtration $\mathcal{A}_n$.
   (b) Find $\{a_n\}_{n\geq 1}$ such that $\eta_n = S_n^3 + a_n S_n$ is a martingale w.r.t filtration $\mathcal{A}_n$.
   (c) Find $\{b_n\}_{n\geq 1}$ such that $\delta_n = S_n^4 + b_n S_n^2 + c_n$ is a martingale w.r.t filtration $\mathcal{A}_n$.
   (d) Let $\{V_k\}_{k\geq 1}$ be a *predictable process*, that is for $c \in \mathbb{R}$,

$$\{V_k = c\} \in \mathcal{A}_{k-1}.$$

   Then show that

$$Z_0 = 0 \qquad Z_n = \sum_{k=1}^{n} V_k(S_k - S_{k-1})$$

   is a martingale w.r.t the filtration $\mathcal{A}_n$.