Due: Wednesday, October 15th, 2025
Time and Problems due: will be mentioned in class

1. Let $\Omega = \{-1, 1\}^{\mathbb{N}}$. Consider for $k \geq 1$, $X_k : \Omega \to \{-1, 1\}$ which are i.i.d. X with distribution given by

$$\mathbb{P}(X = 1) = p = 1 - \mathbb{P}(X = -1),$$

for some $0 < p < 1$.

   (a) For $n \geq 1$, let $S_n : \Omega \to \mathbb{Z}$ be given by $S_n(\omega) = \sum_{k=1}^{n} X_k(\omega)$ and $S_0 = 0$. Show that

$$W_n = \left(\frac{1-p}{p}\right)^{S_n}$$

   is a martingale.

   (b) Let $p = \frac{1}{2}$ and for $n \geq 1$ let

$$Y_n = \frac{1}{2} + \sum_{k=1}^{n} \frac{1}{2^{k+1}} X_k.$$

   Show that Y_n is a martingale. (*Any guess if Y_n converges as $n \to \infty$ and if it does what is its limiting distribution ?*)

   (c) Let $\mathcal{A}_0 = \{\emptyset, \Omega\}$ and $\mathcal{A}_n$ be the observable events by time $n \geq 1$ by $\{X_k\}_{1 \leq k \leq n}$. Let $V_k : \Omega \to \mathbb{R}$ be sequence of bounded discrete random random variables that are predictable, i.e.

$$\{V_k = c\} \in \mathcal{A}_{k-1}, \qquad \text{for all } c \in \text{Range}(V_k), \text{ and } k \geq 1.$$

   Show that $S_n^V = \sum_{k=1}^{n} V_k(S_k - S_{k-1})$, is a martingale, if $p = \frac{1}{2}$.

2. Let $\{\xi_{i,n}\}_{i \geq 1, n \geq 1}$ be a collection of i.i.d. random variables having the same distribution as ξ, with

$$p_k = \mathbb{P}(\xi = k), \text{ for } k \in \mathbb{N} \cup \{0\}, \qquad p_k \geq 0 \qquad \text{and} \qquad \sum_{k=0}^{\infty} p_k = 1.$$

   Let Z_n be the Bienaymé-Galton-Watson process with offspring distribution ξ and starting with one particle at time 1. That is, $Z_1 = 1$ and for $n \geq 2$ let $Z_n = \sum_{i=1}^{Z_{n-1}} \xi_{i,n}$. Let $m = \mathbb{E}[\xi]$

   (a) Show that $\frac{Z_n}{m^n}$ is a martingale.

   (b) If $m < 1$, then using the first moment method, show that $\mathbb{P}(Z_n = 0 \text{ for all large n}) = 1$.

3. We place M marbles in P pots. At each time unit we choose one of the marbles uniformly at random and place it in one of the urns also uniformly chosen at random. Denote by M_n to be the number of marbles in the first urn at time n. Find a_n, b_n, so that $a_n M_n + b_n$ is a martingale.

4. An urn contains R red and G green balls. At each time we draw a ball from the urn, then replace it, and add C balls of the colour drawn. Let $\{X_n\}_{n \geq 1}$ be the fraction of green balls after the n-th draw. Show that X_n is a martingale.