Due: Wednesday, October 22nd, 2025
Time and Problems due: will be mentioned in class

1. Consider the graph $\Gamma = (V, E)$ with $V = \mathbb{Z}^2$ and $E = \{\{\mathbf{i}, \mathbf{j}\} : \mathbf{i}, \mathbf{j} \in V, \mathbf{i} \neq \mathbf{j} \text{ and } \| \mathbf{i} - \mathbf{j} \| \leq \sqrt{2}\}$ with $\| \cdot \|$ denoting the euclidean norm. Consider the standard weight function $\mu \equiv 1$ on E. Let $\{X_n\}_{n\geq 1}$ be the random walk on this weighted graph (Γ, μ) with $X_0 = 0$.
   (a) Show that $M_n = \| X_n \|^2 - \frac{3}{2}n$ for all $n \geq 1$, is a martingale.
   (b) Let $\tau_R = \min\{n \geq 0 : \| X_n \|^2 \geq R^2\}$. Show that
   $$\frac{2}{3}R^2 \leq E[\tau_R] \leq \frac{2}{3}(R + \sqrt{2})^2$$
2. Let $\{Z_n\}_{n\geq 1}$ be a martingale on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and for some integer m, let $Y_n = Z_{n+m} - Z_m$. Show that $\{Y_n\}_{n\geq 1}$ is a martingale.
3. (*Ex 7.21 DSP Gallagher*)Let $\{X_i\}_{i\geq 1}$ be a sequence of i.i.d. random variables such that $\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = \frac{1}{2}$. Let $Z_1 = -2$ and
   $$Z_{n+1} = Z_n(1 + X_n\left(\frac{3n+1}{n+1}\right))$$
   (a) Show that $\{Z_n\}_{n\geq 1}$ is a martingale.
   (b) Let $J = \min\{n \geq 2 : \text{sign}(Z_n) = \text{sign}(Z_{n-1})\}$ be a stopping time. Using induction on n show that
   $$\mathbb{P}(Z_n = \frac{(-2)^n}{n} \mid J > n) = 1 \qquad \text{and } \mathbb{P}(Z_n = -\frac{(-2)^n(n-2)}{n^2 - n} \mid J = n) = 1$$
   (c) Show that $\mathbb{E} \mid Z_J \mid = \infty$ and $\lim_{n\to\infty} \mathbb{E}[Z_n \mid J > n]\mathbb{P}(J > n) = 0$