Due: Wednesday, October 29th, 2025
Time and Problems due: will be mentioned in class

1. Consider a martingale where Z_n can take on only the values 2^{-n-1} and $1-2^{-n-1}$, each with probability $\frac{1}{2}$.
   (a) Given that Z_n, conditional on Z_{n-1}, is independent of $Z_{n-2}, Z_{n-3}, \ldots, Z_1$ find $\mathbb{P}(Z_n = \cdot \mid Z_{n-1})$ for each n so that the martingale condition is satisfied.
   (b) Show that $\mathbb{P}(\sup_{n\geq 1} Z_n \geq 1) = \frac{1}{2} \neq 0 = \mathbb{P}(\bigcup_{n\geq 1}\{Z_n \geq 1\})$
   (c) Show that for all $a > 0$ and $0 < \epsilon < a$, $\mathbb{P}(\sup_{n\geq 1} Z_n \geq a) \leq \frac{E[Z_1]}{a-\epsilon}$.

2. Let $\{X_n\}_{n\geq 1}$ be independent with
$$X_n = \begin{cases} 1 & \text{with probability } \frac{1}{2n} \\ 0 & \text{with probability } 1-\frac{1}{n} \\ -1 & \text{with probability } \frac{1}{2n} \end{cases}$$
For $n \geq 1$,
$$Y_n = \begin{cases} X_n & \text{if } n = 1 \text{ or } Y_{n-1} = 0, n \geq 2 \\ nY_{n-1} \mid X_n \mid & \text{if } Y_{n-1} \neq 0, n \geq 2. \end{cases}$$
   (a) Show that Y_n is a martingale.
   (b) Find $\lim_{n\to\infty} \mathbb{P}(Y_n = 0)$.
   (c) Decide if Martingale convergence theorem applies and whether Y_n converges to some Y as $n \to \infty$ with probability 1.

3. Let $m \in \mathbb{N}$ and $m \geq 2$. At time $n = 1$, an urn contains $2m$ balls, of which m are red and m are blue. At each time $n = 2, 3, \ldots m$ we draw a randomly chosen ball from the urn and record its colour. We do not replace it. Therefore, at time n the urn contains $2m - n + 1$ balls. For $n = 1, \ldots 2m$ let N_n denote the number of red balls and f_n denote the fraction of red balls in the urn after time n.
   (a) Show that f_n is a martingale.
   (b) Let T be the first time at which the ball drawn is red. Find $\mathbb{E}(f_T)$.

4. Let $\{Y_k\}_{k\geq 1}$ be a sequence of discrete random variables. Let $\mathcal{A}_k$ be the observable events by time k. Let X be another random variable such that $E[\mid X \mid] < \infty$. Define $X_n = E[X \mid \mathcal{A}_n]$
   (a) Show that $\{X_n\}_{n\geq 1}$ is a martingale.
   (b) Does X_n converge as $n \to \infty$?

5. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let
$$L_2 = \{X : \Omega \to \mathbb{R} \,|\, \mathbb{E}[X^2] < \infty\}.$$
Suppose $\{\xi_n\}_{n\geq 1}$ and ξ are in L_2 such that $E[(\xi_n - \xi)^2] \to 0$ as $n \to \infty$. Show that $E(\xi_n) \to E(\xi)$ as $n \to \infty$.