Not Due

1. Let $\{X_k\}_{k\geq 1}$ be i.i.d. X with $\mathbb{P}(X=1)=\frac{1}{2}=\mathbb{P}(X=-1)$. Suppose $S_0=0$ and $S_n=\sum_{i=1}^n X_i$

   (a) Let $J=\min\{n\geq 1 : S_n=-a\}$ for $a\in\mathbb{N}\cup\{0\}$ Show that J is a stopping time with respect to $\mathcal{A}_n$–being observable events with regard to $\{X_i\}_{1\leq i\leq n}$.

   (b) For $\theta\in\mathbb{R}, n\geq 1$, let $Z_n=\exp(\theta S_n)(\cosh(\theta))^{-n}$, with $\cosh(\theta)=\frac{1}{2}\left(\exp(\theta)+\exp(-\theta)\right)$

   i. Show that $\{Z_n\}_{n\geq 1}$ is a martingale.

   ii. Show that, when $\theta<0$, the $Z_{\min\{n,J\}}$ is a bounded martingale.

   iii. Let J be as in part(a). Show that[1] for $0<\alpha<1$, one has

   $$\mathbb{E}[\alpha^J]=\left(\frac{1-\sqrt{1-\alpha^2}}{\alpha}\right)^a.$$

2. Let $\{X_k\}_{k\geq 1}$ be a sequence of i.i.d. X such that $p=\mathbb{P}(X=1)=1-\mathbb{P}(X=-1)$ with $0\leq p<\frac{1}{2}$. Let $S_n=\sum_{i=1}^n X_i$.

   (a) Show that for $k\geq 1$, $\mathbb{P}(\bigcup_{n\geq 1}\{S_n\geq k\})=\mathbb{P}(\bigcup_{n\geq 1}\{S_n\geq 1\})^k$

   (b) Let $x=\mathbb{P}(\bigcup_{n\geq 1}\{S_n\geq 1\})$. Show that x solves $x=p+(1-p)x^2$ and find x

   (c) Show that $\frac{p}{1-p}=\exp(-r^\star)$ where $r^\star$ is the unique positive root of $\mathbb{E}[e^{rX}]=1$.

3. (Branching Process) For $n\geq 1$, let $\{\xi_k^n\}_{k\geq 1}$ be i.i.d. ξ with range of X being in $\mathbb{N}\cup\{0\}$, $m=\mathbb{E}[\xi]$ and $\sigma^2=\mathrm{Var}(\xi)$. Define a sequence Z_n by, $Z_1=1$ and for $n\geq 2$,

   $$Z_n=\sum_{i=1}^{Z_{n-1}}\xi_i^n,$$

   with the sum being interpreted as 0 if $Z_{n-1}=0$.

   (a) Show that $\frac{Z_n}{m^n}$ is a martingale.

   (b) Let $m>1$. Find $\mathbb{E}[Z_n]$ and using the law of total variance appropriately find $\mathrm{Var}(Z_n)$. Apply the second moment method show that $\mathbb{P}(Z_n\geq 1\text{for all } n\geq 1)>0$.

   (c) Let $m=1$. Show that there is a random variable Z such that Z_n converges to Z with probability one, as $n\to\infty$. What can you say about the distribution of Z ?

4. Let S_n be the simple symmetric walk on $\mathbb{Z}^d$ starting at 0. Let $h:\mathbb{Z}^d\to\mathbb{R}$ be harmonic, i.e.

   $$h(x)=\frac{1}{2d}\left(\sum_{i=1}^d h(x+e_d)+h(x-e_d)\right),$$

   whenever $x\in\mathbb{Z}^d$, where $\{e_i : 1\leq i\leq d\}$ are the standard basis for $\mathbb{Z}^d$.

   (a) For $d=1,2,3$ provide three examples of harmonic functions h.

   (b) Show that $\{Z_n\}_{n\geq 1}$ given by $Z_n=h(S_n)$ for $n\geq 1$ is a martingale.

[1] You may use the Bounded convergence theorem: namely if $\{X_n\}_{n\geq 1}$ are a sequence of bounded random variables and $X_n\to X$ with probability 1 then $\mathbb{E}[X_n]\to\mathbb{E}[X]$