Due: Wednesday, August 20th, 2025

Time and problems to be turned in: to be announced on August 20th in class

1. Let $S = \{0, 1, 2, 3, 4\}$ and the transition matrix be given by

$$\mathbf{P} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

   Let S_n be a Markov chain on S with transition matrix $\mathbf{P}$ and initial distribution $\mathbb{P}(S_0 = 0) = 1$.

   (a) Draw a weighted graph so that the simple random walk on it would be S_n.

   (b) Find the stationary distribution π of S_n

   (c) Find the $\lim_{n\to\infty} \mathbb{P}(S_n = i)$ for each $i \in S$.

2. Let X_n be a Markov chain on state space $S = \{1, 2, 3, 4, 5, 6, 7\}$ with transition matrix

$$\mathbf{P} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}.$$

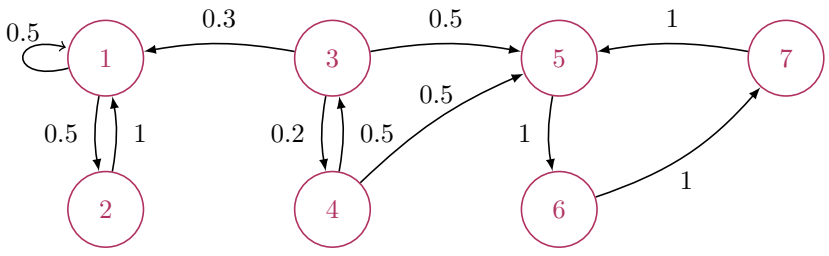
   Assume that $\mathbb{P}(X_0 = 1) = 1$.

   (a) Sketch the graph induced by this Markov chain on the vertex set S.

   (b) Determine the communicating classes of this chain and identify which ones are closed.

   (c) For each communicating class, determine the period of its elements.

3. Assume $\mathbb{P}(X_0 = 1) = 1$. Let X_n be the Markov chain described by the graph below.



   (a) Find the state space and the transition matrix corresponding to this chain.

   (b) Determine the communicating classes of this chain and identify which ones are closed.

(c) For each communicating class, determine the period of its elements.

4. Let $0 < p < 1$, $L \geq 1$ and $S = \{0, 1, 2, \ldots, L\}$. Assume $\mathbb{P}(S_0 = 1) = 1$ and S_n be a Markov chain with transition matrix $\mathbf{P} = [p_{ij}]_{i,j \in \mathbb{Z}}$ given by

$$p_{ij} = \begin{cases} p & \text{if } j = i + 1, i \neq L \\ 1 - p & \text{if } j = i - 1, i \neq 0 \\ p & \text{if } j = 0, i = L \\ 1 - p & \text{if } j = L, i = 0 \\ 0 & \text{otherwise} \end{cases}$$

(a) Sketch the graph induced by this Markov chain on the vertex set S.

(b) Show that the chain is irreducible and determine its periodicity.

(c) Find the stationary distribution π of the chain.

5. Let X_n be a Markov chain on state space S with transition matrix $\mathbf{P}$. Let $i \in S$ have period 1. Show that there is an n_0 such that $p_{ii}^{[n]} > 0$ for all $n \geq n_0$

6. Let $S = \mathbb{Z}$ and transition matrix $\mathbf{P} = [p_{ij}]_{i,j \in \mathbb{Z}}$ be given by

$$p_{ij} = \begin{cases} \frac{1}{2} & \text{if } j = i + 1 \\ \frac{1}{2} & \text{if } j = i - 1 \\ 0 & \text{otherwise} \end{cases}$$

Let S_n be a Markov chain on S with transition matrix $\mathbf{P}$ and initial distribution $\mathbb{P}(S_0 = 0) = 1$. Show that S_n is the simple symmetric random walk considered in HW1-3.