**Due: Thursday, August 28th, 2025**

*Time and problems to be turned in: to be announced on August 28th in class*

1. Let $0 < \alpha, \beta < 1$. Consider a Markov chain $X_n$ on state space $S = \{0, 1\}$, with $P(X_0 = 0) = \mu_0 \in (0, 1), P(X_0 = 1) = \mu_1 \in (0, 1)$, and transition matrix

$$\mathbf{P} = \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix}.$$

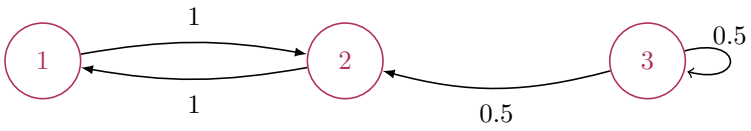
Show that for $n \geq 1$,

$$p_{10}^{[n]} = p_{10}^{[n-1]}(1 - \alpha - \beta) + \beta,$$

and use the technique for solving recurrence relations to show that

$$p_{10}^{[n]} = \frac{\beta}{\alpha + \beta} - \frac{\beta}{\alpha + \beta}(1 - \alpha - \beta)^n.$$

2. Consider a Markov chain $X_n$ on state space $S = \{1, 2, 3\}$, and whose transitions are given via the directed graph




   (a) Find the transition matrix $\mathbf{P}$ for the chain.

   (b) Find the period $d(i)$ for each $i \in S$

   (c) Find the stationary distribution(s) of $X_n$.

   (d) Find the limiting distribution of $X_n$.

3. Let $X_n$ be a Markov chain on $S$ with transition matrix $P$ and initial distribution $\mu$. For $i, j \in S$, if $i \leftrightarrow j$ then show that $d(i) = d(j)$.

4. Let $a, b, c \in \mathbb{R}$ and $D = b^2 - 4ac$. Let $\{x_n\}_{n \geq 1}$ be a sequence such that for $n \geq 2$ it satisfies:

$$ax_n + bx_{n-1} + cx_{n-2} = 0. \tag{1}$$

   (a) Let $D > 0$ and $\alpha, \beta$ be the two distinct roots of the quadratic equation $ax^2 + bx + c$. Show that

$$x_n = A\alpha^n + B\beta^n,$$

   where $A, B \in \mathbb{R}$ is a solution to (1) and any solution to (1) is of this form.

   (b) Let $D = 0$ and $\alpha$ be the real repeated root of the quadratic equation $ax^2 + bx + c$. Show that

$$x_n = (A + nB)\alpha^n$$

   is a solution to (1) and any solution to (1) is of this form.

5. Let $L \in \mathbb{N}$, $X_n$ be a Markov chain on $S = \{0, 1, \dots, L\}$ with transition matrix $\mathbf{P}$ with

$$p_{ij} = \begin{cases} 1 & \text{if } i = j \in \{0, L\} \\ \frac{1}{2} & \text{if } i \geq 1, j = i + 1,\ i \in \{1, \dots, L-1\} \\ \frac{1}{2} & \text{if } i \geq 1, j = i - 1,\ i \in \{1, \dots, L-1\} \\ 0 & \text{otherwise} \end{cases}$$

where we assume $0 < p_i < 1$ for every $i \geq 1$. $h : S \to [0, 1]$ to be given by

$$h(i) = \mathbb{P}(T^{\{0\}} < T^{\{L\}} \mid X_0 = i).$$

Find $h$.

6. Let $X_n$ be a Markov chain on $S = \{0, 1, \dots\}$ with transition matrix $\mathbf{P}$ with

$$p_{ij} = \begin{cases} 1 & \text{if } i = j = 0 \\ p_i & \text{if } i \geq 1, j = i + 1 \\ 1 - p_i & \text{if } i \geq 1, j = i - 1 \\ 0 & \text{otherwise} \end{cases}$$

where we assume $0 < p_i < 1$ for every $i \geq 1$. Let $A = \{0\}$ and let $h^A : S \to [0, 1]$ to be given by

$$h^A(i) = \mathbb{P}(T^A < \infty \mid X_0 = i).$$

Find $h^A$.