Due: Wednesday, September 3rd, 2025

Time and problems to be turned in: to be announced on September 3rd in class

1. Let X_n be a Markov chain on S with transition matrix P and initial distribution μ. Assume that $\mu(\{i\}) > 0$ for all $i \in S$. Let $A \subset S$ and T^A be the hitting time of the set A. For $n \geq 2$ show that

$$\mathbb{P}(T^A = n \mid X_0 = i) = \sum_{j \notin A} p_{ij} \mathbb{P}(T^A = n-1 \mid X_0 = j).$$

2. Let X_n be a Markov chain on S with transition matrix P and initial distribution μ. Assume that the chain is recurrent and $\mu(\{i\}) > 0$ for all $i \in S$. Show that

$$i \in S \text{ is recurrent} \iff \mathbb{P}(T_i < \infty \mid X_0 = j) = 1 \quad \text{for all } j \in S$$

3. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Suppose we have a sequence of events such that $A_n \supseteq A_{n+1}$ for all $n \geq 1$. Show that

$$P(\cap_{n=1}^{\infty} A_n) = \lim_{n \to \infty} P(A_n)$$

4. Let X_n be a Markov chain on S with transition matrix P and initial distribution μ. Assume the chain is irreduc that $\mu(\{i\}) > 0$ for all $i \in S$.
   (a) If there are states $j, k \in S$ are such that $k \leftarrow j$ but $j \not\leftarrow k$, then show that j is a transient state.
   (b) If $i, j \in S$, $i \leftrightarrow j$ and i is recurrent, then j is also recurrent.

5. Let $d \geq 1$. For any $x \in \mathbb{Z}^d$, let $\mid x \mid = \sqrt{\sum_{j=1}^d x_j^2}$. Consider

$$\Omega = \left\{\omega = (\omega_i)_{\imath \geq 1} : \omega_i \in \mathbb{Z}^d \text{ such that } \mid \omega_i \mid = 1\right\}.$$

   Let for any $N \in \mathbb{N}$ define $\pi_N : \Omega \to \Omega_N$ by $\pi_N(\omega) = (\omega_1, \omega_2, \ldots, \omega_N)$ be the projection onto the first N coordinates.. Suppose $(\Omega, \mathcal{F}, \mathbb{P})$ be the probability space where for any fixed $\tilde{\omega} \in \Omega_N$

$$\mathbb{P}(\omega \in \Omega : \pi_N(\omega) = \omega_N) = \frac{1}{(2d)^N}.$$

   For $1 \leq k$,,let $X_K : \Omega \to \{-1, 1\}$ be given by $X_k(\omega) = \omega_k$ and for $1 \leq n$, let $S_n : \Omega \to \{-1, 1\}$ be given by $S_n(\omega) = \sum_{k=1}^n X_k(\omega)$ and $S_0 = 0$.
   (a) Show that $(S_n)_{n \geq 1}$ is a Markov chain on $\mathbb{Z}^d$ and describet its transition matrix.
   (b) Show that for $d = 2$, $(S_n)_{n \geq 1}$ is recurrent.
   (c) Show that for $d = 3$, $(S_n)_{n \geq 1}$ is transient.

6. Consider the Markov chain X_n on $S = \{0, 1, 2, 3, 4, 5\}$ with initial distribution μ and transition matrix

$$\mathbf{P} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

   (a) For each $i \in S$ decide if it is transient or recurrent.
   (b) Describe each closed communicating class.