Due: Wednesday, September 10th, 2025
Time and problems to be turned in: None

1. Let X_n be a Markov chain on S with transition matrix P and initial distribution μ. Find the stationary distribution in each of the following cases.

   (a) (*Reflected Simple Random Walk*) Let $S = \{0, 1, 2, \ldots, m\}$ and the transition matrix $\mathbf{P} = [p_{ij}]_{i,j\in S}$ be given by

   $$p_{ij} = \begin{cases} \frac{1}{2} & \text{if } j = i+1, i \notin \{0, m\} \\ \frac{1}{2} & \text{if } j = i-1, i \notin \{0, m\} \\ 1 & \text{if } j = 1, i = 0 \\ 1 & \text{if } j = m-1, i = m \\ 0 & \text{otherwise} \end{cases}$$

   Describe the chain and any additional feature if we consider the transition matrix $Q = (I+\mathbf{P})/2$?

   (b) (*Simple Random Walk on a Circle*) Let $S = \{0, 1, 2, \ldots, m\}$ and the transition matrix $\mathbf{P} = [p_{ij}]_{i,j\in S}$ be given by

   $$p_{ij} = \begin{cases} \frac{1}{2} & \text{if } j = i+1, i \neq m \\ \frac{1}{2} & \text{if } j = i-1, i \neq 0 \\ \frac{1}{2} & \text{if } j = m, i = 0 \\ \frac{1}{2} & \text{if } j = 0, i = m \\ 0 & \text{otherwise} \end{cases}$$

   Describe the chain and any additional feature if we consider the transition matrix $Q = (I+\mathbf{P})/2$?

2. Let $\mathbb{T}_2$ be a rooted binary tree, with root ρ. That is, a graph with the vertex set given by

   $$\cup_{n=1}^{\infty}\{0,1\}^n \cup \{\rho\}$$

   and edge set given by

   $$\{\{\rho, x\} : x \in \{0,1\}\} \cup \{\{x, p(x)\} : x \in \{0,1\}^n, n \geq 2\},$$

   where for $n \geq 2$, and $x = (x_1, x_2, \ldots, x_n)$ we set the parent $p(x) = (x_1, x_2, \ldots, x_{n-1})$ and $\mid x \mid$ denote the graph distance to the root ρ. On $\mathbb{T}_2$, consider the weight function μ to be given by

   $$\mu(\{x, p(x)\}) = \beta^{|x|} \text{ for } x \in \mathbb{T}_2$$

   where β is a positive number. Let X_n denote the canonical random walk on $(\mathbb{T}_2, \mu)$.

   (a) Show that for $\beta < \frac{1}{2}$ that the random walk X_n is recurrent.

   (b) Consider $Y_n = \mid X_n \mid$.

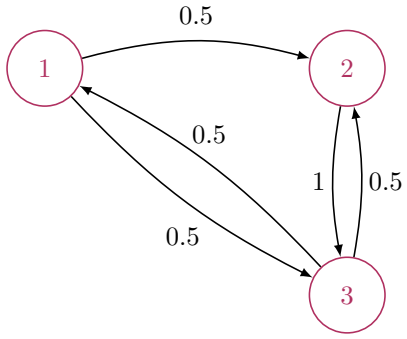
      i. Show that Y_n is a Markov Chain on $\mathbb{Z}_+$.

      ii. Show that Y_n is recurrent when $\beta = \frac{1}{2}$.

iii. Show that Y_n is transient when $\beta > \frac{1}{2}$.

(c) Conclude that X_n is transient if and only if $\beta > \frac{1}{2}$.

3. Consider a Markov chain X_n on state space $S = \{1, 2, 3\}$, with dynamics of the chain given via a directed graph with S as the vertex set.



Find the

$$\lim_{n\to\infty} p_{11}^{[n]} \qquad \lim_{n\to\infty} p_{12}^{[n]} \qquad \lim_{n\to\infty} p_{13}^{[n]}.$$