Due: Wednesday, October 1, 2025
Time and Problems due: will be mentioned in class

1. Let $p_0, p_1, \ldots$ be numbers in $(0,1)$. Let M be a positive integer. Let $\{X_n\}_{n\geq 0}$ be a Markov chain on the state space $S = \{0, 1, \ldots M\}$ with transition matrix $\mathbf{P}$ given by

$$p_{ij} = \begin{cases} p_i & \text{if } j = i+1, i \in \{0,1,\ldots,M-1\} \\ 1-p_i & \text{if } j = i-1, i \in \{1,\ldots,M\} \\ 1-p_0 & \text{if } j = i = 0 \\ p_M & \text{if } i = j = M \\ 0 & \text{otherwise.} \end{cases}$$

   Find the reversible stationary distribution for the chain.

2. Suppose you have d items. For each $i = 1, \ldots d.$ the i-th item worth ₹r_i, and weighs w_i kilograms. You can put some of these items into a bag, but the total weight cannot be more than b kilograms[1]. Let $w = (w_1, w_2, \ldots w_d) \in \mathbb{R}^d$ and let

$$S = \{z \in \{0,1\}^d : \sum_{i=1}^{d} w_i z_i \leq b\}$$

   Let X_0 be any point in S. Suppose we are given X_k for $0 \leq k \leq n$ for some $n \geq 0$ and let $X_k = (z_1, z_2, z_3, \ldots, z_d) \in S$. Construct X_{k+1} via the following:

   (i) Choose U to be a uniformly chosen number in $\{1, 2, 3, \ldots, d\}$.
   (ii) Define $y = (z_1, z_2, \ldots, 1 - z_U, \ldots z_d)$, i.e. flip the U-th coordinate of X_k.
   (iii) If $y \in S$ then set $X_{k+1} = y$ else set $X_{k+1} = X_k$.

   (a) Describe what the algorithm is doing in words.
   (b) What is the stationary distribution of $\{X_n\}_{n\geq 0}$?

3. Let $X \sim$ Binomial (n, p). Using the Chernoff bounds show that for $\delta > 0$

   (a) $P(X \geq (1+\delta)np) \leq \exp(-\frac{\delta^2(np)}{2+\delta})$
   (b) $P(X \leq (1-\delta)np) \leq \exp(-\frac{\delta^2(np)}{2})$

4. At the Indian Statistical Institute to control the pace of walking among students several walk-traffic signals are placed. Suppose the number of red signal Suppandi encounters while coming to class on an average is 6. Suppandi will be late if they encounter 8 or more red signals. Let X be the number of red signals on a given day.

   (a) Give a bound for $P(X \geq 8)$ using Markov inequality.
   (b) Suppose $\text{Var}[X] = 3$, give a bound for $P(X \geq 8)$ using Chebyschev's inequality.
   (c) Suppose $X \sim \text{Binomial}(12, 0.5)$, give a bound for $P(X \geq 8)$ using Chernoff bound.
   (d) Compare and see which one gave the tightest bound.

5. Let $\{A_n\}_{n\geq 1}$ be a sequence of pairwise independent events. Fix $n \geq 1$. For $m > n$, let $X_m = \sum_{i=n}^{m} 1_{A_i}$.

   (a) Show that $P(X_m \geq 1) \geq \frac{1}{1+(\sum_{k=n}^{m} P(A_k))^{-1}}$
   (b) Using (a) show that if $\sum_{k=1}^{\infty} P(A_k) = \infty$ then $P(A_n \text{ occur i.o.}) = 1$

[1]This is known as the Knapsack problem. A key question: What is the most valuable subset of items that will fit into the bag ? It is known to be NP-complete