Introduction to Stochastic Processes **Midterm** **Max Score: 50** Solution
Semester I 2025/26 **Date: 10th September 2025.** **Time:** 1 hour and 30 min

1. (**15 points**) Let X_n be a Markov chain on $S = \{0, 1, \dots\}$ with transition matrix $\mathbf{P}$ with

$$p_{ij} = \begin{cases} 1 & \text{if } i = j = 0 \\ p_i & \text{if } i \geq 1, j = i+1 \\ 1 - p_i & \text{if } i \geq 1, j = i-1 \\ 0 & \text{otherwise} \end{cases}$$

where we assume $0 < p_i < 1$ for every $i \geq 1$. Let $h : S \to [0, 1]$ to be given by

$$h(i) = \mathbb{P}(T^0 < \infty \mid X_0 = i),$$

with $T^0 = \inf\{n \geq 0 : X_n = 0\}$. Find h.

Solutions: Let $q_i = 1 - p_i$. We have that

$$h(i) = \begin{cases} 1 & \text{if } i = 0 \\ \\ p_i h(i+1) + q_i h(i-1) & \text{if } i \geq 1 \end{cases} \tag{1}$$

Let $u_i = h(i-1) - h(i)$. From the above, we have

$$u_{i+1} = \frac{q_i}{p_i} u_i,$$

for $i \geq 1$. Iterating the above, inductively we obtain for all $i \geq 1$

$$u_{i+1} = \left(\prod_{k=1}^{i} \frac{q_k}{p_k} \right) u_1.$$

Using a telescoping sum argument inductively we have that $h(0) - h(i) = \sum_{j=1}^{i} u_j$, and this implies for $i \geq 1$,

$$h(i) = 1 - u_1 \left[\sum_{j=1}^{i} \prod_{k=1}^{j-1} \left(\frac{q_k}{p_k} \right) \right].$$

If

$$\sum_{j=1}^{\infty} \prod_{k=1}^{j-1} \left(\frac{q_k}{p_k} \right) = \infty \tag{2}$$

then as $0 \leq h(i) \leq 1$ for all $i \geq 0$ this will force $u_1 = 0$ which will imply $u_i = 0$ for all $i \geq 1$ and consequently

$$h(i) = 1 \text{ for all } i \geq 0. \tag{3}$$

If

$$\sum_{j=1}^{\infty} \prod_{k=1}^{j-1} \left(\frac{q_k}{p_k} \right) < \infty \tag{4}$$

then to satisfy the requirement that $0 \leq h(i) \leq 1$ it must be that

$$0 \leq u_1 \leq \frac{1}{\sum_{j=1}^{\infty} \prod_{k=1}^{j-1} \left(\frac{q_k}{p_k}\right)}.$$

and so by Proposition 1.5.5 (b) this implies that

$$u_1 = \frac{1}{\sum_{j=1}^{\infty} \prod_{k=1}^{j-1} \left(\frac{q_k}{p_k}\right)}.$$

Therefore for $i \geq 1$

$$h(i) = \frac{\sum_{j=i+1}^{\infty} \prod_{k=1}^{j-1} \left(\frac{q_k}{p_k}\right)}{\sum_{j=1}^{\infty} \prod_{k=1}^{j-1} \left(\frac{q_k}{p_k}\right)}. \tag{5}$$

□

2. (**15 points**) Let $\{S_n\}_{n\geq 0}$ be a simple symmetric random walk on $\mathbb{Z}$ starting at 0.

   (a) For $a < b, a, b \in \mathbb{Z}$, $n \geq 1$ show that

   $$\mathbb{P}(a \leq S_n \leq b) \leq (b - a + 1)\mathbb{P}(S_n \in \{0, 1\})$$

   (b) $\lim_{n\to\infty} \mathbb{P}(a \leq S_n \leq b) = 0.$

Solutions: (a) Let $n \geq 2$. By definition of randm walk $S_n \in \{-n, -n+2 \ldots, n-2, n\}$ and by defintion of $\mathbb{P}$ we have

$$\mathbb{P}(S_n = l) = \begin{cases} \binom{n}{(n+l)/2} 2^{-n} & l \in \{-n, -n+2 \ldots, n-2, n\} \\ 0 & \text{otherwise} \end{cases}.$$

Thus, maximizing $\mathbb{P}(S_n = l)$ over $l \in \mathbb{Z}$ amounts to maximizing the binomial coefficient $\binom{n}{(n+l)/2}$ over $l \in \{-n, -n+2, \ldots, n-2, n\}$. This maximum occurs at $\frac{n+l}{2} = \frac{n}{2}$, i.e. $a = 0$ when n is even, and at $\frac{n+l}{2} = \frac{n+1}{2}$, i.e. $l = 1$ when n is odd. So we obtain

$$\max\{\mathbb{P}(S_n = l) : l \in \mathbb{Z}\} = \begin{cases} \mathbb{P}(S_{2k} = 0) & \text{if } n = 2k, k \in \mathbb{N} \\ \mathbb{P}(S_{2k-1} = 1) & \text{if } n = 2k-1, k \in \mathbb{N} \end{cases}$$

Therefore it is immediate

$$\mathbb{P}(a \leq S_n \leq b) = \sum_{l \in [a,b]} \mathbb{P}(a \leq S_n = l) \leq (b - a + 1)\mathbb{P}(S_n \in \{0, 1\})$$

(b) Now from the above,

$$\mathbb{P}(S_n \in \{0, 1\}) = \begin{cases} \binom{n}{n/2} 2^{-n} & \text{when } n \text{ is even} \\ \\ \binom{n+1}{(n+1)/2} 2^{-(n+1)} & \text{when } n \text{ is odd.} \end{cases}$$

Stirling's approximation for $n!$, implies that

$$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n}(\frac{n}{e})^n} = 1 \tag{6}$$

From the definition of limit in (6) there exists $N_0 \in \mathbb{N}$ and such that

$$\frac{1}{2}\sqrt{2\pi n}(\frac{n}{e})^n \leq n! \leq \frac{3}{2}\sqrt{2\pi n}(\frac{n}{e})^n \text{ for all } n \geq N_0.$$

Using the above, we have that for $n \geq N_0$

$$\mathbb{P}(S_{2n} = 0) = \frac{(2n)!}{n!n!}\frac{1}{2^{2n}} \leq \frac{6\sqrt{\pi n}(2n)^{2n}e^{-2n+1}}{(\sqrt{2\pi n}n^n e^{-n})^2 2^{2n}} = \frac{3e}{\sqrt{\pi}}\frac{1}{\sqrt{n}}.$$

and

$$\begin{aligned}
\mathbb{P}(S_{2n+1} = 0) &= \frac{(2(n+1))!}{n+1!n+1!}\frac{1}{2^{2(n+1)}} \\
&\leq \frac{6\sqrt{\pi(n+1)}(2(n+1))^{2(n+1)}e^{-2(n+1)+1}}{(\sqrt{2\pi(n+1)}(n+1)^{(n+1)}e^{-(n+1)})^2 2^{2(n+1)}} \\
&= \frac{3e}{\sqrt{\pi}}\frac{1}{\sqrt{n+1}} \\
&< \frac{3e}{\sqrt{\pi}}\frac{1}{\sqrt{n}}
\end{aligned}$$

Thus for all $n \geq N_0$, there exist constant $C_1 > 0$ such that

$$0 \leq \mathbb{P}(S_n \in \{0, 1\}) \leq \frac{C_1}{\sqrt{n}}.$$

Using this and part(a) we have that

$$0 \leq \mathbb{P}(a \leq S_n \leq b) \leq \frac{(b-a+1)C_1}{\sqrt{n}}.$$

As $\frac{(b-a+1)C_1}{\sqrt{n}} \to 0$ when $n \to \infty$ we have

$$\lim_{n\to\infty} \mathbb{P}(a \leq S_n \leq b) = 0.$$

□

3. (**15 points**) Let $0 < p < 1$, $L \geq 1$ and $S = \{0, 1, 2, \ldots, L\}$. Assume $\mathbb{P}(S_0 = 1) = 1$ and S_n be a Markov chain with transition matrix $\mathbf{P} = [p_{ij}]_{i,j\in\mathbb{Z}}$ given by

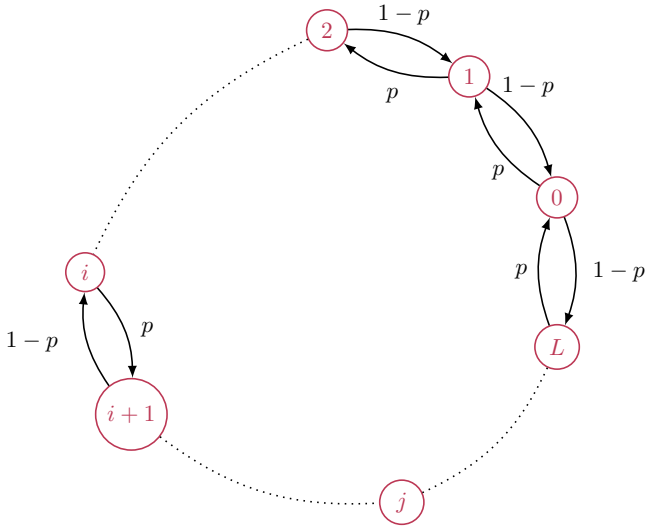
$$p_{ij} = \begin{cases} p & \text{if } j = i+1, i \neq L \\ 1-p & \text{if } j = i-1, i \neq 0 \\ p & \text{if } j = 0, i = L \\ 1-p & \text{if } j = L, i = 0 \\ 0 & \text{otherwise} \end{cases}$$

(a) Sketch the graph induced by this Markov chain on the vertex set S.

(b) Show that the chain is irreducible and determine its periodicity.

(c) Find the stationary distribution π of the chain.

Solutions: (a) The above dynamics of the chain can be understood via a directed graph with S as the vertex set.



(b) Let $0 \leq i < j \leq L$ and $m = j - i$. Then,

$$p_{ij}^m > \prod_{k=1}^{m} p_{i+k-1,i+k} = p^m > 0.$$

Let $0 \leq j < i \leq L$ and $m = i - j$. Then,

$$p_{ij}^m > \prod_{k=1}^{m} p_{i-k+1,i-k} = (1-p)^m > 0.$$

Therefore the chain is irreducible.

(c) Observe that for any $i, j \in S$,

$$\sum_{i=0}^{L} p_{ij} = \sum_{j=0}^{L} p_{ij} = p + 1 - p = 1.$$

Therefore the transition matrix is doubly stochastic. This implies that the stationary distribution is Uniform$(\{0, 1, \ldots, L\})$. □

4. (**15 points**) Let $\mathbb{T}_2$ be a rooted binary tree, with root ρ. That is, a graph with the vertex set given by

$$\cup_{n=1}^{\infty} \{0,1\}^n \cup \{\rho\}$$

and edge set given by

$$\{\{\rho, x\} : x \in \{0,1\}\} \cup \{\{x, p(x)\} : x \in \{0,1\}^n, n \geq 2\},$$

where for $n \geq 2$, and $x = (x_1, x_2, \ldots, x_n)$ we set the parent $p(x) = (x_1, x_2, \ldots, x_{n-1})$. On $\mathbb{T}_2$, consider the weight function μ to be given by

$$\mu(\{x, p(x)\}) = \beta^{|x|} \text{ for } x \in \mathbb{T}_2$$

where β is a positive number and $\mid x \mid$ denote the graph distance to the root ρ. Let X_n denote the canonical random walk on $(\mathbb{T}_2, \mu)$. Show that X_n is transient if and only if $\beta > \frac{1}{2}$.

Solutions: Let S be the vertex set of $\mathbb{T}_2$. Let $S_n = \{x \in S : |x| = n\}$. It can be readily verified that $S = \cup_{n \geq 0} S_n$, where the union is disjoint and using an inductive argument that $|S_n| = 2^n$ for all $n \geq 1$. Now

$$\text{for } x \in S, \qquad \mu_x = \sum_{x \sim y, x, y \in S} \mu(\{x, y\}).$$

Note that X_n is a Markov chain on S with transition matrix P where $P = [p_{xy}]_{x,y \in S}$ is given by

$$p_{xy} = \frac{\mu(\{x,y\})}{\mu_x} = \begin{cases} \frac{1}{2} & \text{if } x = p(y), x = \rho \\ \frac{1}{1+2\beta} & \text{if } y = p(x), x \in S, x \neq \rho \\ \frac{\beta}{1+2\beta} & \text{if } x = p(y), x \in S, x \neq \rho \\ 0 & \text{otherwise.} \end{cases}$$

We will show that $Y_n = \mid X_n \mid$ is a simple random walk on $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. First, note that Y_n takes values in $\mathbb{Z}_+$ and that $Y_n = y$ if and only if $X_n \in S_y$ for any $n, y \in \mathbb{N}$. Suppose $y, n \in \mathbb{Z}_+$, we have $\mathbb{P}[Y_n = y] > 0$. Then , for $z \in \mathbb{Z}_+$:

$$\begin{aligned} \mathbb{P}[Y_{n+1} = z | Y_n = y] &= \mathbb{P}[X_{n+1} \in S_z | X_n \in S_y] \\ &= \frac{1}{\mathbb{P}[X_n \in S_y]} \sum_{u \in S_z} \mathbb{P}[X_{n+1} = u, X_n \in S_y] \\ &= \frac{1}{\mathbb{P}[X_n \in S_y]} \sum_{u \in S_z, v \in S_y} \mathbb{P}[X_{n+1} = u, X_n = v] \\ &= \frac{1}{\mathbb{P}[X_n \in S_y]} \sum_{u \in S_z, v \in S_y} p_{uv} \mathbb{P}[X_n = v] \\ &= \begin{cases} \frac{1}{1+2\beta} & \text{if } z = y - 1, y \neq 0 \\ \frac{2\beta}{1+2\beta} & \text{if } z = y + 1, z \neq 0 \\ 0 & \text{otherwise.} \end{cases} \\ &:= q_{yz} \end{aligned} \tag{7}$$

Suppose for $y_j \in \mathbb{Z}_+$ with $j = 0, 1, \ldots, n$ we have $\mathbb{P}[Y_n = y_n, \ldots, Y_0 = y_0] > 0$, then for

$y_{n+1} \in \mathbb{Z}_+$

$$\mathbb{P}[Y_{n+1} = y_{n+1} | Y_n = y_n, \ldots, Y_0 = y_0] = \mathbb{P}[X_{n+1} \in S_{y_{n+1}} | X_n \in S_{y_n}, \ldots, X_0 \in S_{y_0}]$$
$$= \sum_{x_{n+1} \in S_{y_{n+1}}} \mathbb{P}[X_{n+1} = x_{n+1} | X_n \in S_{y_n}, \ldots, X_0 \in S_{x_0}]$$

using the Markov Property for the Markov chain $\{X_n\}_{n \geq 1}$ we have that the above is

$$= \sum_{x_i \in S_{y_i}, 1 \leq i \leq n+1} \mathbb{P}[X_{n+1} = x_{n+1} | X_n = x_n, \ldots, X_0 = x_0] \mathbb{P}[X_n = x_n, X_{n-1} = x_{n-1}, \ldots, X_0 = x_0]$$
$$= \sum_{x_i \in S_{y_i}, 1 \leq i \leq n+1} \mathbb{P}[X_{n+1} = x_{n+1} | X_n = x_n] \mathbb{P}[X_n = x_n, X_{n-1} = x_{n-1}, \ldots, X_0 = x_0]$$
$$= \sum_{x_{n+1} \in S_{y_{n+1}}} \mathbb{P}[X_{n+1} = x_{n+1} | X_n \in S_{y_n}]$$
$$= \mathbb{P}[Y_{n+1} = y_{n+1} | Y_n = y_n]$$

Inductively (and arguing similarly for case $\mathbb{P}[Y_{n+1} = y_{n+1}, Y_n = y_n, \ldots, Y_0 = y_0] = 0$) we have

$$\mathbb{P}[Y_{n+1} = y_{n+1}, Y_n = y_n, \ldots, Y_0 = y_0] = \mathbb{P}(Y_0 = y_0) \prod_{i=1}^{n} q_{y_{i+1} y_i}$$

Thus we have show that Y_n is a Markov chain on $\mathbb{Z}_+$ with transition matrix $Q = [q_{yz}]_{y,z \in \mathbb{Z}_+}$.

Let $h : S \to [0, 1]$ be such that $h(i) = \mathbb{P}_i(T_Y^{\{0\}} < \infty)) = 1$, where $T_Y^{\{0\}} = \min\{n \geq 0 : Y_n = 0\}$. Note that h solves (1) with $p_i = \frac{2\beta}{1+2\beta}$ and $\frac{q_i}{p_i} = \frac{1}{2\beta}$

Let $\beta \leq \frac{1}{2}$. Then (2) holds and for all $i \geq 0$, $h(i) = 1$ from (3). Therefore 0 is a recurrent state which implies that ρ is a recurrent state for X_n. As X_n is irreducible we have that X_n is recurrent on S.

Let $\beta > \frac{1}{2}$. Then (4) holds and for all $i \geq 1$, $h(i) < 1$ from (5). Therefore 0 is a transient state which implies that ρ is a transient state for X_n. As X_n is irreducible we have that X_n is transient on S.