1. (*Monkey typing "ISP IS BORING"*) A monkey draws independently and uniformly from the English alphabet $S = \{A, B, C, D, \dots Z\}$. Let U_i be the result of the i-th draw. Let $\mathcal{A}_n$ be the observable events from $U_1, U_2, \dots, U_n$. Let the random time τ be defined as

$$\tau = \min\{n \geq 11 : (U_{n-10}, U_{n-9}, \dots U_n) = (I, S, P, I, S, B, O, R, I, N, G)\}$$

   (a) Show that τ is a stopping time and $\mathbb{E}[\tau] < \infty$

   (b) Find a martingale $(M_n)_{n\geq 1}$ such that

      i. $M_1 = 0$ and $M_\tau = \tau - 26^{11}$

      ii. there is a $C > 0$ such that $\mid M_{n+1} - M_n \mid \leq C$ for all $n \geq 1$

   (c) Show that $\mathbb{E}[\tau] = 26^{11}$

Using Markov chain

$\tilde{A}$: (Markov chain)

$Y_1 = (U_1, U_2, \dots, U_{11})$

$Y_2 = (U_2, \dots, U_{12})$

$\vdots$

$Y_k = (U_k, \dots, U_{k+10})$

$Y_k \in S^{11}$

Set $Y_0 = (A\,B\,C\,D\,E\,F\,G\,H\,I\,J\,K)$

Ex: $\{Y_k\}_{k\geq 1}$ is a Markov chain on S^{11}

$P \equiv$ can be identified.

Find $E_{Y_0}[\tau] = ?$

Set $h(y) = E_y(\tau)$

.. Ex .. h is a minimal solution to ...

$h((I,S,P,I,S,B,O,R,I,N,G)) = 0$

and $h(y) = \sum_{z \in S} p_{yz} h(z)$, $y \neq (I,S,P,I,S,B,O,R,I,N,G)$

Solving ... By hand is hard. □

(a) $\{\tau = k\} = \{(U_{k-10}, U_{k-9}, \dots, U_k) = (I,S,P,I,S,B,O,R,I,N,G)\}$

$\bigcap_{j=11}^{k-9} \{(U_{j-10}, U_{j-9}, \dots, U_j) \neq (I,S,P,I,S,B,O,R,I,N,G)\}$

$\therefore \{\tau = k\}$ is observable from $U_1, \dots, U_k$.

So its a stopping time.

(b) Use Martingales

Let $(\psi_1, \psi_2, \ldots \psi_{10}, \psi_{11}) = (I, S, P, I, S, B, O, R, N, G)$

Casino :-

1 2 3 ..... time

$\{Z'_k\}_{k \geq 1}$ = capital of gambler arrives at $t=1$

$\boxed{Z'_0 = 1}$ $t=1$

places a bet 1 Rs & bets $U_1 = I$

If $U_1 = I$ then gambler gets 26 Rs

If $U_1 \neq I$ then gambler gets 0 Rs and exits casino.

Fair game :- Expected Profit : $\frac{1}{26}(26-1) + \frac{25}{26}(-1) = 0$

$t=2$ If $Z'_1 = 26$ & Gambler bets all 26 Rs that $U_2 = S$

If $U_2 = S$ then gambler gets $(26)^2$ Rs

If $U_2 \neq S$ then gambler gets 0 Rs and exits casino.

and so on till $t=11$ & $Z'_t = Z'_{11}$ $\forall t \geq 10$

$$\therefore \quad Z_n^1 = \begin{cases} 0 & n=0 \\ Z_{n-1}^1(26)\,\mathbb{1}(U_n=\psi_n) & 1\le n\le 11 \\ Z_{11}^1 & n\ge 12 \end{cases}$$

Now a 2nd gambler arrives at $t=2$ again with initial capital 1Rs. places a bet that $U_2 = I$ & gets 26 if it happens & 0 otherwise. She continues till time 12 similarly.

For any $k\ge 1$, let gambler k enter the casino start at $t=k$ similarly. So,

let $$Z_n^k = \begin{cases} 1 & n<k \\ 26\, Z_{n-1}^k \mathbb{1}(U_n=\psi_{n-k+1}) & k\le n\le k+10 \\ Z_{k+10}^k & n\ge k+11 \end{cases}$$

Net Profit of casino from k^{th} Gamble $=$

$$= 1 - Z_n^k$$

Let $$M_n = \sum_{k=1}^{n}(1-Z_n^k) = n - \sum_{k=1}^{n} Z_n^k$$

- $M_n = n - \sum_{k=1}^{n} Z_n^k \quad \Rightarrow \quad |M_n| < n + (26)^n$

$\Rightarrow \quad E|M_n| < \infty$

- $E[M_n \mid U_{n-1}, \dots, U_1]$

$= E\left[n - \sum_{k=1}^{n} Z_n^k \mid U_{n-1}, \dots, U_1 \right]$

$= n - E\left[\sum_{k=1}^{n-9} Z_n^k + \sum_{k=n-10}^{n} Z_k^n \mid U_{n-1}, .., U_1 \right]$

By definition $= n - E\left[\sum_{k=1}^{n-9} Z_{n-1}^k + \sum_{k=n-10}^{n} 26\, Z_{n-1}^k \mathbb{1}_{(U_n = \Psi_{n-k+1})} \mid U_{n-1}, .., U_1 \right]$

U_n is independent of $U_{n-1}, \dots, U_1$ $= n - \left(\sum_{k=1}^{n-9} Z_{n-1}^k + \sum_{k=n-10}^{n} 26\, Z_{n-1}^k \, \mathbb{P}(U_n = \Psi_{n-k+1}) \right)$

Ex $= n - 1 - \sum_{k=1}^{n-1} Z_{n-1}^k$

$\Rightarrow \{M_n\}_{n \geq 1}$ is a Martingale.

(i) $M_1 = 0$ &

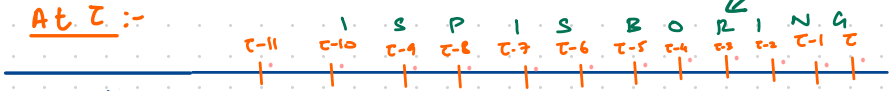
Now $M_n = n - \sum_{k=1}^{n} Z_n^k$

$M_\tau = \tau - \sum_{k=1}^{\tau} Z_\tau^k$

$\{Z_n^{\tau-10}\}_{n\geq 1}$

$Z_\tau^{\tau-10} = (26)^{11}$

At τ :-



As there is no repetition of initial part of string

$Z_\tau^k = 0$ for $\tau-10 < k \leq \tau$ &

$\Rightarrow \quad M_\tau = \tau - (26)^{11}$

(ii) Now for $n \geq 2$,

$|M_n - M_{n-1}| = \left| \sum_{k=1}^{n-1} (Z_{n-1}^k - Z_n^k) - Z_n^n + 1 \right|$

if $n > k+10$, $Z_n^k - Z_{n-1}^k = 0$ and $Z_n^n \leq 26$

$\leq \left| \sum_{k=n-10}^{n-1} (Z_{n-1}^k - Z_n^k) \right| + 25$

$= \left| \sum_{k=n-10}^{n-1} (26\, 1(U_n = \psi_{n-k+1}) - 1) Z_{n-1}^k \right| + 25$

$\leq 25 \sum_{k=n-10}^{n-1} |Z_{n-1}^k| + 25$

$Z_k^n \leq 25^{11}$ $\leq (25)^{12} (n-1-(n-10)+1) + 25$

$\leq (25)^{12} \cdot 10 + 25 := c$

(c) Let for $n \geq 1$, divide the sequence

$u_1 \; u_2 \; u_{11} \; .. \; .. \; .. \; u_k \; .. \; .. \; u_n \; ..$

w_1 w_2

into blocks of size 11.

Let S be the number of blocks until a block $\equiv (\psi_1, .., \psi_{11})$

Then $S \sim$ Geometric $\left(\frac{1}{26^{11}}\right)$

and $\tau \leq 11 S$ (Ex.)

$\Rightarrow$ $E\tau < 11 \, E[S] = 11(26)^{11} < \infty$ —⊕

$\therefore$ By ⊕ and (b)

OST hypothesis are satisfied

By OST : $E[M_\tau] = E[M]$ —⊗

$E[M_1] = E[1 - Z_1'] = 1 - \frac{26}{26} + \frac{25}{26}(0) = 0$

$E[M_\tau] = 0 \overset{(b)(i)}{\Rightarrow} E[\tau - (26)^{11}] = 0$

$\Rightarrow E[\tau] = (26)^{11}$ □

2. (*Erdös Renyi Graph- Threshold for isolated vertices*) Let $n \geq 1$, $0 < \lambda < n$ and $V = \{1, 2, \ldots, n\}$ be the vertex set. Consider the Erdös Renyi Graph $G(n, \frac{\lambda}{n})$ in which we place an edge between $i, j \in V$ with probability $\frac{\lambda}{n}$. Define

$$Y_n = \sum_{i=1}^{n} 1(\text{ vertex } i \text{ is a isolated vertex in } G(n, \frac{\lambda}{n}))$$

to be the number of isolated vertices in $G(n, \frac{\lambda}{n})$.

(a) Show that $\lambda \geq \frac{1}{2}$)

$$n \exp(-\lambda) \exp(\frac{-\lambda^2}{n}) \leq \mathbb{E}[Y_n] \leq n \exp(-\lambda) \exp(\frac{\lambda}{n}).$$

(b) Show that

$$\text{Var}(Y) \leq \mathbb{E}[Y_n] + \frac{\lambda}{\lambda - n}(\mathbb{E}[Y])^2.$$

(c) Show that if $\lambda - \log(n) \to -\infty$ then $\mathbb{P}(Y_n = 0) \to 0$ as $n \to \infty$.

(d) Show that if $\lambda - \log(n) \to \infty$ then $\mathbb{P}(Y_n = 0) \to 1$ as $n \to \infty$.

(a) $\mathbb{E}[Y_n] = \sum_{i=1}^{n} \mathbb{P}(1(\text{ vertex } i \text{ is a isolated vertex in } G(n, \frac{\lambda}{n})))$

Ex $= n\,\mathbb{P}(1(\text{ vertex } 1 \text{ is a isolated vertex in } G(n, \frac{\lambda}{n})))$

$= n(1 - \frac{\lambda}{n})^{n-1}$

Ex : $e^{x - x^2} \leq 1 - x \leq e^{-x}$ $\quad 0 \leq x \leq \frac{1}{2}$

$\Rightarrow$ result $\square$

(b) $\mathbb{E}[Y_n^2] \overset{\text{Ex}}{=} n\,\mathbb{P}(1(\text{ vertex } 1 \text{ is a isolated vertex in } G(n, \frac{\lambda}{n})))$

$+ n(n-1)\,\mathbb{P}(1(\text{ vertex } 1 \text{ is a isolated vertex in } G(n, \frac{\lambda}{n}))\ 1(\text{ vertex } 2 \text{ is a isolated vertex in } G(n, \frac{\lambda}{n})))$

$\text{Var}[Y_n] = \mathbb{E}[Y_n^2] - (\mathbb{E}[Y_n])^2$

$= \mathbb{E}[Y_n^2] - (n\,\mathbb{P}(1(\text{ vertex } 1 \text{ is a isolated vertex in } G(n, \frac{\lambda}{n}))))^2$

$= n\,\mathbb{P}(1(\text{ vertex } 1 \text{ is a isolated vertex in } G(n, \frac{\lambda}{n})))$

$+ n(n-1)\,\mathbb{P}(1(\text{ vertex } 1 \text{ is a isolated vertex in } G(n, \frac{\lambda}{n}))\ 1(\text{ vertex } 2 \text{ is a isolated vertex in } G(n, \frac{\lambda}{n})))$

$- n^2 (\mathbb{P}(1(\text{ vertex } 1 \text{ is a isolated vertex in } G(n, \frac{\lambda}{n}))))^2$

$$= n\left[\mathbb{P}\left(1(\text{vertex } 1 \text{ is a isolated vertex in } G(n,\tfrac{\lambda}{n}))\right) - \mathbb{P}\left(1(\text{vertex } 1 \text{ is a isolated vertex in } G(n,\tfrac{\lambda}{n}))\ 1(\text{vertex } 2 \text{ is a isolated vertex in } G(n,\tfrac{\lambda}{n}))\right)\right]$$

$$+ n^2\left[\mathbb{P}\left(1(\text{vertex } 1 \text{ is a isolated vertex in } G(n,\tfrac{\lambda}{n}))\ 1(\text{vertex } 2 \text{ is a isolated vertex in } G(n,\tfrac{\lambda}{n}))\right) - \left(\mathbb{P}\left(1(\text{vertex } 1 \text{ is a isolated vertex in } G(n,\tfrac{\lambda}{n}))\right)\right)^2\right]$$

Using $\mathbb{P}\left(1(\text{vertex } 1 \text{ is a isolated vertex in } G(n,\tfrac{\lambda}{n}))\ 1(\text{vertex } 2 \text{ is a isolated vertex in } G(n,\tfrac{\lambda}{n}))\right) = \left(1-\frac{\lambda}{n}\right)^{2n-2}$

(Ex)

$$\mathrm{Var}(Y_n) \le \mathbb{E}[Y_n] + \frac{\lambda}{n-\lambda}\left(\mathbb{E}[Y_n]\right)^2 \qquad \square$$

(c) Suppose $\lambda - \log n \longrightarrow -\infty$ — (*)

$$0 \le \mathbb{E}[Y] \le n e^{-\lambda} e^{\lambda/n}$$

and $n e^{-\lambda} e^{\lambda/n} \longrightarrow 0$ by (*)

$$\Rightarrow \mathbb{E}[Y_n] \longrightarrow 0 \text{ as } n \to \infty.$$

2nd moment method

$$\mathbb{P}(Y_n = 0) \overset{\text{Ex}}{\le} \frac{\mathbb{E}[Y] + \frac{\lambda}{n-\lambda}\left(\mathbb{E}[Y]\right)^2}{\left(\mathbb{E}[Y]\right)^2}$$

$$= \frac{2}{\mathbb{E}[Y]} + \frac{\lambda}{n-\lambda}$$

$$\frac{2}{\mathbb{E}[Y]} + \frac{\lambda}{n-\lambda} \longrightarrow 0 \text{ as } n \to \infty \quad \text{by (*)}$$

by (*)

$\therefore \quad \lambda - \log n \to -\infty \implies P(Y_n = 0) \to 0 \quad -(+)$
as $n \to \infty$. □

(d)

• If $\lambda - \log n \to \infty$, $\lambda \leq 2 \log n \quad -(\pm)$

$$P(Y=0) = 1 - P(Y \geq 1) \geq 1 - E[Y]$$
(Markov)
$$\geq 1 - n e^{-\lambda} e^{\lambda/n}$$

And

$1 - n e^{-\lambda} e^{\lambda/n} \longrightarrow 1$ as $n \to \infty$ by $(\pm)$

□

We will now prove the result below.

Theorem :- (Threshold for connectivity)

(i) If $\lambda - \log n \to -\infty$

$P\left(\text{Erdös-Renyi}\left(n, \frac{\lambda}{n}\right) \text{ is connected}\right) \to 0$

ii) If $\lambda - \log n \to \infty$

$P\left(\text{Erdös-Renyi}\left(n, \frac{\lambda}{n}\right) \text{ is connected}\right) \to 1$

Proof :- (i)

Now $P\left(\text{Erdös-Renyi}\left(n, \frac{\lambda}{n}\right) \text{ is connected}\right)$

$$= \mathbb{P}(Y_n = 0) - \mathbb{P}\left(\text{Erdös-Renyi}\left(n, \frac{\lambda}{n}\right), Y_n = 0\right)$$
$$\text{disconnected}$$

$$\leq \mathbb{P}(Y_n = 0)$$

$\therefore$ If $\lambda - \log n \to -\infty$ by part (c)

$$\Rightarrow \mathbb{P}\left(\text{Erdös-Renyi}\left(n, \frac{\lambda}{n}\right) \text{ is connected}\right) \to 0$$
$$\text{as } n \to \infty$$

□

(ii) We will need the following claim

Claim 2: ("No isolated $\Rightarrow$ connectivity")

There is $a > \frac{1}{2}$, s.t. if $\lambda \geq a \log(n)$ for $a \geq \frac{1}{2}$ then

$$\mathbb{P}\left(\text{Erdös-Renyi}\left(n, \frac{\lambda}{n}\right), Y_n = 0\right) \longrightarrow 0$$
$$\text{disconnected}$$

$$\text{as } n \to \infty$$

Assume claim. As before

$$\mathbb{P}\left(\text{Erdös-Renyi}\left(n, \frac{\lambda}{n}\right) \text{ is connected}\right)$$

$$= \mathbb{P}(Y_n = 0) - \mathbb{P}\left(\text{Erdös-Renyi}\left(n, \frac{\lambda}{n}\right), Y_n = 0\right)$$
$$\text{disconnected}$$

If $\lambda - \log n \to \infty$ then by part (d)

$$\mathbb{P}(Y_n = 0) \longrightarrow 1 \text{ as } n \to \infty$$

and by claim 2

$$P\left(\text{Erdös-Renyi}\left(n, \frac{\lambda}{n}\right), Y_n = 0\right) \longrightarrow 0$$

n disconnected

as $n \to \infty$

$\therefore$ If $\lambda - \log n \to \infty$

$$P\left(\text{Erdös-Renyi}\left(n, \frac{\lambda}{n}\right) \text{ is connected}\right) \longrightarrow 1$$

as $n \to \infty$. □

Sketch of proof Claim 2

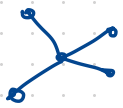
Take a realisation of Erdös-Renyi $\left(n, \frac{\lambda}{n}\right)$

- $X_k \equiv$ # trees of size k which cannot be extended to $k+1$

  i.e $\{v_1, v_2, \ldots, v_k\} \subseteq [1, \ldots, n]$

  contain a tree in Erdös-Renyi $\left(n, \frac{\lambda}{n}\right)$

  & $v \notin \{v_1, v_2, \ldots, v_k\}$ there is no edge between $v \to v_i$ $i = 1, \ldots, k$.

E.g   $n = 8$

$\Rightarrow$ $X_3 = 3$ $X_5 = 1$

Suppose Erdös-Renyi $(n, \frac{\lambda}{n})$ is disconnected

but $Y_1 = 0$ $\exists\ k \in \{2, \dots, \frac{n}{2}\}$ for

$X_k \geq 1$.

$$\mathbb{P}\left(\text{Erdös-Renyi } (n, \tfrac{\lambda}{n}),\ Y_n = 0\right)$$
$$\text{disconnected}$$

$$\leq \mathbb{P}\left(\bigcup_{k=2}^{n/2} \{X_k \geq 1\}\right)$$

$$\leq \sum_{k=2}^{n/2} \mathbb{P}(X_k \geq 1)$$

$$\leq \sum_{k=2}^{n/2} E[X_k]$$

Ex: $E[X_k] = \binom{n}{k} k^{k-2} \cdot \left(\frac{\lambda}{n}\right)^{k-1} \left(1 - \frac{\lambda}{n}\right)^{k(n-k)}$

If $\exists\ a > \frac{1}{2}$ $\lambda \geq a \log n$

then $E[X_k] \leq n^{1 - k/4}$ $k \geq 5$

$\therefore$ $\mathbb{P}\left(\text{Erdös-Renyi } (n, \tfrac{\lambda}{n}),\ Y_n = 0\right)$
$$\text{disconnected}$$

$$\leq 4n(e\lambda)^4 e^{-\lambda k} + \sum_{k=5}^{n/2} n^{1-k/4} \cdot \longrightarrow 0 \quad \text{as } n \to \infty$$
(Ex)

□

3. (*Large Deviation*) Let $\{X_k\}_{k\geq 1}$ be i.i.d Bernoulli $(\frac{1}{2})$ and $S_n = \sum_{k=1}^n X_k$. Show that for $a > \frac{1}{2}$,

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(S_n \geq an) = -I(a),$$

where

$$I(a) = \begin{cases} \log(2) + a\log(a) + (1-a)\log(a) & a \in [0,1] \\ \infty & \text{otherwise.} \end{cases}$$

Using Symmetry show that the same conclusion holds for $\mathbb{P}(S_n \leq an)$ for $a < \frac{1}{2}$. Conclude the SLLN for $\{X_k\}_{k\geq 1}$ using the Borel-Cantelli lemma.

See: Large Deviations by Frank den Hollander
Section 1.2

□

4. (*Random Walk in Trap model*) Let $\{X_n\}_{n\geq 1}$ be a simple random walk on $\mathbb{Z}^d$ with $X_0 = 0$. Let $\{N_y\}_{y\in\mathbb{Z}^d}$ be i.i.d. Poisson random variables with mean ν, and let $\{Y_n^{y,j}\}_{n\geq 1, y\in\mathbb{Z}^d, 1\leq j\leq N_y}$ be a family of independent simple symmetric random walks on $\mathbb{Z}^d$ where $Y^{y,j}$ denotes the j-th random walk starting from y at time 0. Let the number of traps at site x at time n be

$$\xi(n,x) := \sum_{y\in\mathbb{Z}^d, 1\leq j\leq N_y} \delta_x(Y_n^{y,j}).$$

Fix $0 \leq q \leq 1$, which will be the trapping probability. At each time $n \geq 0$, X is killed with probability $1-(1-q)^{\xi(n,X_n)}$. Namely, each trap at the time-space lattice site $(n, \bar{X}(n))$ tries independently to capture X with probability q.

(a) In $d = 1$ draw a pictorial representation of the above model.

(b) Given a realization of X, let $\sigma^X(n)$ denote the probability that X has survived till time n. Show that

$$\sigma^X(n) = \mathbb{E}^\xi\left[(1-q)^{\sum_{i=0}^n \xi(i,X(i))}\right] = \exp\Big\{-\nu\sum_{y\in\mathbb{Z}^d} w^{q,X}(n,y)\Big\},$$

where

$$w^{q,X}(n,y) := 1 - \mathbb{E}_y^Y\left[(1-q)^{\sum_{i=0}^n 1_{\{Y_i=X_i\}}}\right],$$

where under $\mathbb{E}_y^Y$, Y is a simple symmetric random walk starting at y.

(c) *Pascal Principle:* it is known that given a realization of X,

$$\sigma^X(n) \leq \sigma^{\mathbf{0}}(n),$$

where $\mathbf{0}$ is the trajectory of the walk X that stays at 0 all the time.

i. Explain the significance of Pascal Principle and describe it in words.

ii. For a proof see Lemma 2.2 in https://arxiv.org/abs/1010.3958.

See : Section 2 in