3. Martingale - Discrete time & Discrete Space

- Historically: Gambling systems in 18th Century

Preliminaries :-

- Conditional Expectation: Suppose X and Y are discrete random variables

$y \in \text{Range}(Y) \subseteq \mathbb{R}^d$ i.e. $\mathbb{P}(Y=y) > 0$

$$E[X \mid Y=y] = \sum_{k \in \text{Range}(X) \subseteq \mathbb{R}^d,\ d \geq 1} k\, \mathbb{P}(X=k \mid Y=y)$$

If Range(X) is countably infinite :- the series should converge for well-definedness.

Definition 1:- let $(\Omega, \mathcal{F}, \mathbb{P})$ be a Probability space and $\{Z_n : n \geq 1\}$ be a sequence of random variables on it. $\{Z_n\}_{n \geq 1}$ is a martingale if

- $E|Z_n| < \infty \quad \forall n \geq 1.$
- $E[Z_n \mid Z_{n-1} = z_{n-1}, Z_{n-2} = z_{n-2}, \ldots, Z_1 = z_1] = z_{n-1}$

whenever $\mathbb{P}(Z_{n-1} = z_{n-1}, Z_{n-2} = z_{n-2}, \ldots, Z_1 = z_1) > 0, \forall n \geq 2.$

Example 1: $S_n = \sum_{i=1}^{n} X_i$ $\qquad X_i = \begin{cases} 1 & \text{w.p } \frac{1}{2} \\ -1 & \text{w.p } \frac{1}{2} \end{cases}$

Let $n \geq 2$.

Suppose $\mathbb{P}(S_{n-1} = s_{n-1}, S_{n-2} = s_{n-2}, \dots, S_1 = s_1) > 0$ for some $\{s_i\}_{i \geq 1}$ $s_i \in \mathbb{Z}$.

- $\mathbb{E}[S_n \mid S_{n-1} = s_{n-1}, S_{n-2} = s_{n-2}, \dots, S_1 = s_1]$

$$= \sum_{k \in \mathbb{Z}} k \, \mathbb{P}(S_n = k \mid S_{n-1} = s_{n-1}, S_{n-2} = s_{n-2}, \dots, S_1 = s_1)$$

$$= \sum_{k \in \mathbb{Z}} k \, \mathbb{P}(S_n = k \mid S_{n-1} = s_{n-1})$$

[Already seen: $\{S_n\}_{n \geq 1}$ is a Markov Chain]

$$= \sum_{k \in \mathbb{Z}} k \, \mathbb{P}(X_n = k - s_{n-1} \mid S_{n-1} = s_{n-1})$$

$\uparrow$ $(S_n = X_n + S_{n-1})$

$$= \sum_{k \in \mathbb{Z}} k \, \mathbb{P}(X_n = k - s_{n-1})$$

{ X_n is independent of S_{n-1}

$$= (s_{n-1} + 1)\frac{1}{2} + (s_{n-1} - 1)\frac{1}{2} = s_{n-1}$$

In these calculation all sums are finite sums.

- $\mathbb{E}|S_n| = \mathbb{E}\left|\sum_{k=1}^{n} X_i\right| \leq \sum_{k=1}^{n} \mathbb{E}|X_i| = n < \infty$

$\Rightarrow$ $\{S_n\}_{n \geq 1}$ is a Martingale.

Example 2: On $(\Omega, \mathcal{F}, \mathbb{P})$ let $\{X_k\}_{k \geq 1}$ be random variables

(a) $X_k = \begin{cases} 1 & \text{w.p } p \\ -1 & \text{w.p } 1-p \end{cases}$, $p \neq \frac{1}{2}$ & $S_n = \sum_{k=1}^{n} X_k$

Then $E[S_n \mid S_{n-1} = s_{n-1}, S_{n-2} = s_{n-2}, \ldots, S_1 = s_1]$

$(Ex) = p(s_{n-1} + 1) + (s_{n-1} - 1)(1-p)$

$\neq s_{n-1}$ $(p \neq \frac{1}{2})$

(b) $\{S_n\}_{n \geq 1}$ is a Markov chain but not a martingale.

$X_k = \begin{cases} 1 & \text{w.p } \frac{1}{2} \\ -1 & \text{w.p } \frac{1}{2} \end{cases}$ $(i.i.d.)$ let $Z_1 = \begin{cases} 1 & \text{w.p } \frac{1}{2} \\ 0 & \text{w.p } \frac{1}{2} \end{cases}$

$Z_n = Z_{n-1} + X_n \cdot Z_1$

Ex:- $\{Z_n\}_{n \geq 1}$ is Not a Markov chain.

$E[Z_n \mid Z_{n-1} = z_{n-1}, \ldots, Z_1 = z_1] = z_{n-1}$

as $E[X_n] = 0$.

Example (Product Martingale)

On $(\Omega, \mathcal{F}, \mathbb{P})$, $\{X_n\}_{n \geq 1}$ are a sequence of discrete random variables i.i.d $E[X_1] = 1$.

Let $Z_n = \prod_{i=1}^{n} X_i \quad \forall\, n \geq 1$

let $z_1, \ldots, z_{n-1} \in \mathbb{R}$ s.t

$$\mathbb{P}(Z_{n-1} = z_{n-1}, \ldots, Z_1 = z_1) > 0$$

$$E[Z_n \mid Z_{n-1} = z_{n-1}, \ldots, Z_1 = z_1]$$

$$= \sum_{k \in \text{Range}(Z_n)} k \, \mathbb{P}(Z_n = k \mid Z_{n-1} = z_{n-1}, \ldots, Z_1 = z)$$

$$\overset{Ex}{=} \sum_{k \in \text{Range}(Z_n)} k \, \frac{\mathbb{P}(Z_n = k, Z_{n-1} = z_{n-1}, \ldots, Z_1 = z)}{\mathbb{P}(Z_{n-1} = z_{n-1}, \ldots, Z_1 = z)}$$

$$\overset{Ex}{=} \sum_{k \in \text{Range}(Z_n)} k \, \frac{\mathbb{P}(X_n z_{n-1} = k, Z_{n-1} = z_{n-1}, \ldots, Z_1 = z)}{\mathbb{P}(Z_{n-1} = z_{n-1}, \ldots, Z_1 = z)}$$

$$= \sum_{k \in \text{Range}(Z_n)} k \; \mathbb{P}(X_n z_{n-1} = k) \; \frac{\mathbb{P}(Z_{n-1} = z_{n-1}, \cdots, Z_1 = z)}{\mathbb{P}(Z_{n-1} = z_{n-1}, \cdots, Z_1 = z)}$$

(X_n is independent of $Z_1, \ldots, Z_{n-1}$)

$$\overset{(X_n \overset{d}{=} X_1)}{=} \sum_{k \in \text{Range}(Z_{n-1})} k \; \mathbb{P}(X_1 z_{n-1} = k)$$

$$\overset{Ex}{=} \sum_{u \in \text{Range}(X_1)} u \, z_{n-1} \, \mathbb{P}(X_1 = u)$$

$$\overset{Ex}{=} z_{n-1} \, E[X_1] = z_{n-1}$$

- $E|Z_n| = E\left|\prod_{i=1}^{n} X_i\right| = E\prod_{i=1}^{n}|X_i| = [E|X_1|]^n < \infty.$

$\therefore \{Z_n\}_{n \geq 1}$ is a Martingale.

Further suppose, $X_i \overset{d}{=} X = \begin{cases} 2 & \text{w.p } \frac{1}{2} \\ 0 & \text{w.p } \frac{1}{2} \end{cases}$

$E[X_k] = 1 \quad \forall \; k \geq 1$ & $Z_n = \prod_{i=1}^{n} X_i$ is a martingale

$$P(Z_n = 2^n) = \frac{1}{2^n} \leftarrow$$ high values with low probability

$$P(Z_n = 0) = 1 - \frac{1}{2^n} \leftarrow$$ low values with high probability

Preliminaries :

Meaning of :

$$E[Z_n \mid Z_{n-1} = z_{n-1}, Z_{n-2} = z_{n-2}, \dots, Z_1 = z_1] = z_{n-1}, \quad -(*)$$

whenever $P(Z_{n-1} = z_{n-1}, Z_{n-2} = z_{n-2}, \dots, Z_1 = z_1) > 0$

LHS of * : $E[Z_n \mid Z_{n-1} = z_{n-1}, Z_{n-2} = z_{n-2}, \dots, Z_1 = z_1]$

$\Downarrow$

$f(z_{n-1}, z_{n-2}, \dots, z_1)$

RHS of * : $z_{n-1} := g(z_{n-1})$

Let $Y_{n-1} = f(Z_{n-1}, Z_{n-2}, \dots, Z_1)$ (random variable!)

(using LHS (*) we can find Y_{n-1})

Interpretation :-

Y_{n-1} - can be thought of - "$E[Z_n \mid Z_{n-1}, \dots, Z_1]$"

Re phrase: $\{Z_n\}_{n\geq 1}$ is a Martingale

if

⊕ $\begin{cases} \cdot\ E|Z_n| < \infty, \quad \forall n \geq 1 \\ \cdot\ E[Z_n \mid Z_{n-1}, \ldots, Z_1] = Z_{n-1} \quad \forall n \geq 2 \end{cases}$ and

$E[Z_n \mid Z_{n-1}, \ldots, Z_1] \equiv$ function of $Z_{n-1}, \ldots, Z_1$ is a function of Z_{n-1} if $\{Z_n\}_{n\geq 1}$ is a martingale

Example 1 (revisited) :-

$S_n = \sum_{i=1}^{n} X_i \qquad X_i = \begin{cases} 1 & w.p\ \frac{1}{2} \\ -1 & w.p\ \frac{1}{2} \end{cases}$ (independent)

· Show $\{S_n\}_{n\geq 1}$ is a Martingale using ⊕

· $E|S_n| \leq \sum_{i=1}^{n} E|X_i| \leq n < \infty$

· $E[S_n \mid S_{n-1}, \ldots, S_1]$

$= E[X_n + S_{n-1} \mid S_{n-1}, \ldots, S_1]$

Ex: (linearity of conditional expectation)

$$= E[X_n \mid S_{n-1}, \ldots, S_1] + E[S_{n-1} \mid S_{n-1}, \ldots, S_1]$$

$$= E[X_n] + S_{n-1}$$

- X and Y are r.v.'s such that X is independent of Y

then

$$E[X \mid Y=y] = \sum_{k \in \text{Range}(X)} k \, \mathbb{P}(X=k \mid Y=y)$$

$$= \sum_{k \in \text{Range}(X)} k \, \mathbb{P}(X=k)$$

$$= E[X]$$

$E[X \mid Y] = E[X]$ if X is independent of Y

- Let X, Y, Z be discrete r.v.s $X = f(Y, Z)$ for some f.

$$E[X \mid Y=y, Z=z] = \sum_{k \in \text{Range}(X)} k \, \mathbb{P}(X=k \mid Y=y, Z=z)$$

$$= \sum_{k \in \text{Range}(X)} k \, \mathbb{P}(f(Y,Z)=k \mid Y=y, Z=z)$$

Suppose $u = f(y,z)$

$$= u \, \frac{\mathbb{P}(f(Y,Z)=u, Y=y, Z=z)}{\mathbb{P}(Y=y, Z=z)}$$

$$= u = f(y,z)$$

$E[X \mid Y, Z] = X$ whenever $X = f(Y,Z)$ for some function f

Example 2 (Revisited): $E[X_i] = 1$ & $Z_n = \prod_{i=1}^{n} X_i$

X_i are i.i.d.

$$E[Z_n \mid Z_{n-1}, \dots, Z_1] = E[X_n Z_{n-1} \mid Z_{n-1}, \dots, Z_1)$$

X, Y are discrete r.v.; $(y \neq 0)$

$$E[XY \mid Y=y]$$

$$= \sum_{k \in \text{Range}(XY)} k \, P(XY = k \mid Y=y)$$

$$= \sum_{k \in \text{Range}(XY)} k \, P\left(X = \frac{k}{y} \mid Y=y\right)$$

$$= \sum_{u \in \text{Range}(X)} uy \, P(X=u \mid Y=y)$$

$$= y \sum_{u \in \text{Range}(X)} u P(X=u \mid Y=y)$$

$$= y \, E[X \mid Y=y)$$

$$\boxed{E[XY \mid Y] = Y E[X \mid Y]}$$

$$= Z_{n-1} E[X_n \mid Z_{n-1}, \dots, Z_1)$$

$$= Z_{n-1} E[X_n]$$

↑ X_n is independent of $Z_{n-1}, \dots Z_1$

$$= Z_{n-1}$$

$$X_i \equiv X = \begin{cases} 2 & \text{w.p } \frac{1}{2} \\ 0 & \text{w.p } \frac{1}{2} \end{cases} \Rightarrow P(Z_n = 2^n) = \frac{1}{2^n}$$

$$P(Z_n = 0) = 1 - \frac{1}{2^n}$$

Note:- $\Rightarrow \begin{cases} Z_n \xrightarrow{P} 0 & \text{as } n \to \infty \\ E Z_n \longrightarrow 1 & \text{as } n \to \infty \end{cases}$

Example 4: (Exponential Martingale)

On $(\Omega, \mathcal{F}, \mathbb{P})$ $\quad X_i = \begin{cases} 1 & \text{wp } \frac{1}{2} \\ -1 & \text{wp } \frac{1}{2} \end{cases}$

$S_n = \sum_{i=1}^{n} X_i \qquad S_0 = 0. \qquad \{S_n\}_{n \geq 1}$ - martingale

$$E[e^{rX_i}] = \frac{e^r + e^{-r}}{2} \qquad \forall\, r \in \mathbb{R}.$$

Let $g(r) = \ln E[e^{rX_i}]$

let $r \in \mathbb{R}$ be fixed.

$$Z_n = e^{rS_n - n g(r)} \qquad \forall\, n \geq 1$$

$$= \prod_{i=1}^{n} \left(e^{rX_i - g(r)} \right)$$

Set $Y_i = e^{rX_i - g(r)}$; Y_i are i.i.d.

$\Leftarrow E[Y_i] = 1 \quad \& \quad Z_n = \prod_{i=1}^{n} Y_i$

$\Rightarrow \{Z_n\}_{n \geq 1}$ is a martingale $\square$

(Example 2)

lemma 1 Let $\{Z_n : n \geq 1\}$ be a martingale. Then

$$1 \leq i \leq n, \quad E[Z_n \mid Z_i, .., Z_1] = Z_i \quad - (\ddagger)$$

Proof:-

Claim: [Tower Property] X, Y, Z are discrete random variables then

$$E[\, E[X \mid Y, Z] \mid Y] = E[X \mid Y]$$

Assume claim.

Proof by induction:

$n = i+1$ Then $(\ddagger)$ reduces to

$$E[Z_n \mid Z_i, .., Z_1] = E[Z_{i+1} \mid Z_i, .., Z_1]$$
$$= Z_i \quad \text{(Martingale)}$$

$\therefore$ $(\ddagger)$ is verified.

Assume $n = i+k$ $(\ddagger)$ is true

$\Rightarrow$ $E[Z_{i+k} \mid Z_i, .., Z_1] = Z_i$

$n = i + k + 1$

$$E\left[Z_{i+k+1} \mid Z_i, Z_{i-1}, \ldots, Z_1 \right]$$

Tower Property $=$

$$E\left[E\left[Z_{i+k+1} \mid Z_{i+k}, \ldots, Z_{i+1}, Z_i, Z_{i-1}, \ldots, Z_1 \right] \mid Z_i, Z_{i-1}, \ldots, Z_1 \right]$$

$=$ (Martingale)

$$E\left[Z_{i+k} \mid Z_i, Z_{i-1}, \ldots, Z_1 \right]$$

$=$ (induction)

$$Z_i$$

□

Suppose $\{Z_n\}_{n \geq 1}$ is a Martingale then

Lemma 1 $\Rightarrow$ $E[Z_n \mid Z_1] = Z_1 \quad \forall \ n \geq 1$.

Take expectation on both sides

$$E\left[E\left[Z_n \mid Z_1 \right] \right] = E[Z_1]$$

- $E\left[E[X \mid Y] \right] = E[X]$

$\Rightarrow$ $E[Z_n] = E[Z_1] \quad \forall \ n \geq 1$ □

Expectations stay constant.