Optional stopping theorem

Recall:- Given a sequence $\{Z_n\}_{n\geq 1}$ and a stopping time $T : \Omega \to \mathbb{N} \cup \{\infty\}$ then the stopped process

$\{Z_n^T\}_{n\geq 1}$ is a martingale

whenever

$\{Z_n\}_{n\geq 1}$ is a martingale.

i.e.

- $E[Z_n^T] < \infty$
- $E[Z_n^T \mid Z_{n-1}^T, \dots, Z_1^T] = Z_{n-1}^T$ —(*)

when $\{Z_n\}_{n\geq 1}$ is a martingale.

Example: $X_i = \begin{cases} 2 & \text{w.p. } \frac{1}{2} \\ 0 & \text{w.p. } \frac{1}{2} \end{cases}$ independent

$\forall\, n \geq 1$ let $Z_n = \prod_{i=1}^{n} X_i$

$\Rightarrow \mathbb{P}(Z_n = 2^n) = \frac{1}{2^n}$ and $\mathbb{P}(Z_n = 0) = 1 - \frac{1}{2^n}$

$\forall\, n \geq 2 \quad E[Z_n \mid Z_{n-1}, \dots Z_1] = Z_{n-1}$

$T = \min\{k \geq 1 \mid Z_k = 0\} \Rightarrow Z_T = 0$

Z_n^T – martingale $\Rightarrow$

$E[Z_n^T] = E[Z_1^T] = E[Z_1] = 1$

$Z_n^T \equiv Z_{n \wedge T} \longrightarrow Z_T$ as $n \to \infty$ w.p. 1

& $E[Z_T] = 0$

$\therefore \quad E[Z_n^T] \not\longrightarrow E[Z_T]$ as $n \to \infty$

Theorem 3:- [OST]

T is a stopping time of a martingale $\{Z_n\}_{n \geq 1}$

& $T: \Omega \to \mathbb{N}$.

$E[Z_T] = E[Z_1] \iff$

$E[Z_n | T \geq n] \, P(T \geq n) \longrightarrow 0$ as $n \to \infty$

& $E|Z_T| < \infty$

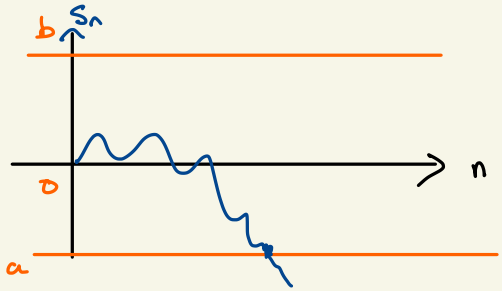
Application of OST :

Let $i \geq 1$ and $X_i = \begin{cases} 1 & \text{w.p } p \\ -1 & \text{w.p } 1-p \end{cases}$ be independent.

$0 < p < 1$

$S_n = \sum_{i=1}^{n} X_i \quad \forall\, n \geq 1$

$S_0 = 0$



$a < 0 < b$

$\sigma_a = \min \{ n \geq 1 \mid S_n = a \}$

$\sigma_b = \min \{ n \geq 1 \mid S_n = b \}$

Q: $P(\sigma_a < \sigma_b) = ?$

A:-

Ex:

- $T = \sigma_a \wedge \sigma_b$ then T is a stopping time
- $P(T \geq N) \longrightarrow 0$ as $N \to \infty$.

HW: $W_n = \left(\frac{1-p}{p}\right)^{S_n}$ is a martingale

Apply OST to $\{W_n\}_{n\geq 1}$. Need to show:

- $E[W_n | T \geq n]\, P(T \geq n) \longrightarrow 0$ as $n \to \infty$ - (*)
  and $E|W_T| < \infty$

Proof of (*):-

$$W_n = \left(\frac{1-p}{p}\right)^{S_n}$$

$$|W_n| \leq \left(\frac{1-p}{p}\right)^{a} + \left(\frac{1-p}{p}\right)^{b} \quad \text{if } T \geq n$$

$\Rightarrow$ (Ex) $\left|E(W_n | T \geq n)\right| \leq \left(\frac{1-p}{p}\right)^{a} + \left(\frac{1-p}{p}\right)^{b}$ - (†)

By Ex $P(T \geq n) \to 0$ as $n \to \infty$ (‡)

$\therefore$ (†) and (‡) $\Rightarrow$ (*)

$\therefore$ By (*) & OST we have

$$E[W_T] = E[W_1]$$

$$= \left(\frac{1-p}{p}\right) P(S_1 = 1) + \left(\frac{p}{1-p}\right) P(S_1 = -1)$$

$$= \left(\frac{1-p}{p}\right) p + \left(\frac{p}{1-p}\right)(1-p) = 1$$

□

But $E[W_T] = E(W_T ; \sigma_a < \sigma_b)$

$+ E(W_T ; \sigma_b \leq \sigma_a)$

$T = \sigma_a \wedge \sigma_b = E(W_{\sigma_a} ; \sigma_a < \sigma_b)$

$+ E(W_{\sigma_b} ; \sigma_b \leq \sigma_a)$

$S_{\sigma_a} = a$, $S_{\sigma_b} = b$ $= \left(\frac{1-p}{p}\right)^a \mathbb{P}(\sigma_a < \sigma_b)$

$+ \left(\frac{1-p}{p}\right)^b \mathbb{P}(\sigma_b \leq \sigma_a)$

$\Rightarrow \quad 1 = \left(\frac{1-p}{p}\right)^a \mathbb{P}(\sigma_a < \sigma_b) + \left(\frac{1-p}{p}\right)^b \mathbb{P}(\sigma_b \leq \sigma_a)$

$\Rightarrow \quad 1 = \left(\frac{1-p}{p}\right)^a \mathbb{P}(\sigma_a < \sigma_b) + \left(\frac{1-p}{p}\right)^b (1 - \mathbb{P}(\sigma_a < \sigma_b))$

$\Rightarrow$

$$\mathbb{P}(\sigma_a < \sigma_b) = \frac{\left(\frac{1-p}{p}\right)^b - 1}{\left(\frac{1-p}{p}\right)^b - \left(\frac{1-p}{p}\right)^a}$$

$a < 0 < b$

Proof of Theorem 3

- $\{Z_n^T\}_{n\geq 1}$ is a martingale

(*) $\Rightarrow E|Z_n^T| < \infty$ & $E[Z_n^T] = E[Z_1]$

$Z_n^T \longrightarrow Z_T$ as $n \to \infty$ w.p. 1

Now: $$E[Z_n^T] = \sum_{i=1}^{n-1} E[Z_n^T : T=i] + E[Z_n^T : T \geq n]$$

(Suitably interpreted if $P(T=i)=0$ for some i)

$$= \sum_{i=1}^{n-1} E[Z_n^T \mid T=i]\, P(T=i) + E[Z_n^T \mid T \geq n]\, P(T \geq n)$$

$$= \sum_{i=1}^{n-1} E[Z_i \mid T=i]\, P(T=i) + E[Z_n^T \mid T \geq n]\, P(T \geq n)$$

Using (*) and above

$$E[Z_1] = \sum_{i=1}^{n-1} E[Z_i \mid T=i]\, P(T=i) + E[Z_n \mid T \geq n]\, P(T \geq n) \quad \text{(**)}$$

Now as $E[Z_T] < \infty$,

$$E[Z_T] \overset{Ex}{=} \sum_{i=1}^{\infty} E[Z_T \mid T=i]\, P(T=i)$$

(Suitably interpreted if $P(T=i)=0$ for some i)

$$\overset{Ex}{=} \sum_{i=1}^{\infty} E[Z_i \mid T=i]\, P(T=i) \quad -(***)$$

Taking limits in (**) if

$$E[Z_n^* \mid T \geq n]\, P(T>n) \longrightarrow 0 \quad \text{as } n \to \infty$$

then $E[Z_T] = E[Z_1]$ ∎.

Checking assumptions of (OST):

- (Ex) If $T : \Omega \to \mathbb{N}$ and $\sup_{\omega \in \Omega} |T(\omega)| < M$

  then clearly assumption of (OST) holds

- (Ex) $P(T<\infty) = 1$ (i.e. $P(T>n) \to 0$ as $n \to \infty$)

  $(Z_n)_{n \geq 1}$ is a bounded martingale

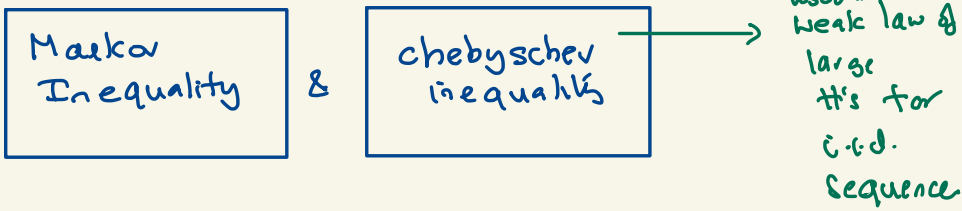
  then clearly assumption of (OST) holds.

- (Hw) $E[T] < \infty$ and $\exists\, C > 0$ $|Z_n - Z_{n-1}| < C$ $\forall\, n \geq 2$

  then assumption of (OST) holds

Kolmogorov Maximal Inequalities

These inequalities are more powerful than :



Kolmogorov Maximal Inequalities used in

- strong law of large numbers
- Martingale convergence theorem.
- Doob's inequality

Theorem 1: Let $\{Z_n\}_{n \geq 1}$ be a martingale such that $E[Z_n^2] < \infty \quad \forall\, n \geq 1$

Then $\forall\, m \geq 1$ and $b > 0$

$$P\left(\max_{1 \leq i \leq m} |Z_i| \geq b \right) \leq \frac{E[Z_m^2]}{b^2}$$

Preliminaries:

Suppose $h: \mathbb{R} \to \mathbb{R}$ is a convex function and $\{Z_n\}_{n \geq 1}$ is a martingale & $E|h(Z_n)| < \infty$ $\forall n \geq 1$

then $E[h(Z_n) \mid Z_{n-1}, .. Z_1]$

$\overset{\text{(Jensen)}}{\geq} h(E(Z_n \mid Z_{n-1}, \cdot, Z_1))$

$= h(Z_{n-1})$

we say $\{h(Z_n)\}_{n \geq 1}$ is a sub-martingale!

Definition: $\{Z_n\}_{n \geq 1}$ a sequence of random variables is said to be a sub-martingale if $E|Z_n| < \infty$ and

$E[Z_n \mid Z_{n-1}, Z_{n-2}, \cdots, Z_1] \geq Z_{n-1}$ $\forall n \geq 2$.

Example: $\{Z_n\}_{n \geq 1}$ be a martingale

let $h(x) = x^2$ and $EZ_n^2 < \infty$

$\Rightarrow$ $\{h(Z_n)\}_{n \geq 1}$ is a sub-martingale

let $h(x) = e^{\alpha x}$ and $E e^{\alpha Z_n} < \infty$

$\Rightarrow \{h(Z_n)\}_{n \geq 1}$ is a sub-martingale

Theorem 1' (Kolmogorov inequality)

Let $\{Z_n\}_{n \geq 1}$ be a non-negative submartingale such

Then $\forall\, m \geq 1$ and $a > 0$

$$P\left(\max_{1 \leq i \leq m} Z_i \geq a \right) \leq \frac{E[Z_m]}{a}$$

Proof of Theorem 1

Observe if $\{Z_n\}_{n \geq 1}$ is a martingale $\Rightarrow \{Z_n^2\}_{n \geq 1}$ is a sub martingale

Then take $a = b^2$ in Theorem 1' to $\{Z_n^2\}_{n \geq 1}$ obtain the result $\square$

Proof of Theorem 1' :-

We need to understand $\rightarrow P\left(\max_{1 \leq i \leq m} Z_i \geq a \right)$

let $T = \min\{k \geq 1 \mid Z_k \geq a\} \wedge m$

If $Z_n < a \quad \forall \; n < m \implies J = m$!

$\overset{(Ex)}{\implies} \quad Z_J \geq a$ iff $Z_n \geq a$ for some $n \leq m$

$$\therefore \quad \mathbb{P}\left(\max_{1 \leq i \leq n} Z_i \geq a \right) = \mathbb{P}\left(Z_J \geq a \right)$$

Markov Inequality

$$\leq \frac{E[Z_J]}{a} \quad - (\star)$$

$$E[Z_J] = E\left[\sum_{k=1}^{m-1} Z_k \mathbb{1}_{J=k} + Z_m \mathbb{1}_{J \geq m} \right]$$

$$= \sum_{k=1}^{m-1} E[Z_k \mathbb{1}_{J=k}] + E[Z_m \mathbb{1}_{J \geq m}]$$

A_k - observable by events $Z_1, \ldots, Z_k$ & $E[Z_m | A_m] = Z_k$

$$= \sum_{k=1}^{m-1} E\left[E[Z_m | A_k] \mathbb{1}_{J=k} \right] + E[Z_m \mathbb{1}_{J \geq m}]$$

$E[XY|A_Y] = Y E[X|A_Y]$

$$= \sum_{k=1}^{m-1} E\left[E[Z_m \mathbb{1}_{J=k} | A_k] \right] + E[Z_m \mathbb{1}_{J \geq m}]$$

- J is stopping time

$$E[\,E[X|Y]\,] = E[X]$$

$$= \sum_{k=1}^{m-1} E[\,Z_m \mathbb{1}_{J=k}\,] + E[Z_m \mathbb{1}_{J\geq m}]$$

$$= E\left[Z_m \left[\sum_{k=1}^{m-1} \mathbb{1}_{J=k} + \mathbb{1}_{J\geq m}\right]\right]$$

$$= E[Z_m] \quad - (*)$$

$(\circledast)$ in $(*)$ $\Rightarrow$

$$\mathbb{P}\left(\max_{1\leq i\leq m} Z_i \geq a\right) \leq \frac{E\,Z_m}{a}$$

□

Theorem 2: Let $\{Z_n\}_{n\geq 1}$ be a non-negative martingale. Then $\forall a > 0$

$$\mathbb{P}\left(\sup_{n\geq 1} Z_n \geq a\right) \leq \frac{E[Z_1]}{a}$$

Proof:- In particular $\{Z_n\}_{n\geq 1}$ is a non-negative submartingale.

$\forall\, m \geq 1$ $\quad \mathbb{P}\left(\max_{1\leq i\leq m} Z_i \geq a\right) \leq \frac{E\,Z_m}{a} = \frac{E\,Z_1}{a}$

↑ Martingale

If $A_m = \left\{\max_{1\leq i\leq m} Z_i \geq a\right\}$ $\Rightarrow$ $A_m \subseteq A_{m+1}$ $\forall\, m \geq 1$

$$A_m \uparrow A = \bigcup_{i \geq 1} \{Z_i \geq a\} \quad \text{(key)}$$

let $\varepsilon_0 > 0$ be given & so that $0 < \varepsilon_0 < a$

$$P\left(\sup_{n \geq 1} Z_n \geq a\right) = P\left(\bigcap_{\varepsilon > 0} \bigcup_{i=1}^{\infty} \{Z_i \geq a - \varepsilon\}\right)$$

$$\leq P\left(\bigcup_{i=1}^{\infty} \{Z_i \geq a - \varepsilon_0\}\right) \quad - (*)$$

Apply Theorem 1' to $b = a - \varepsilon_0$ to get

$$P\left(\max_{1 \leq i \leq m} Z_i \geq a - \varepsilon_0\right) \leq \frac{E[Z_1]}{a - \varepsilon_0} \quad - (+)$$

$$B_m^{\varepsilon_0} = \left\{\max_{1 \leq i \leq m} Z_i \geq a - \varepsilon_0\right\} \uparrow \bigcup_{i \geq 1} \{Z_i \geq a - \varepsilon_0\} \quad (++)$$

$\therefore$ (+) and (++)

$$P\left(\bigcup_{i \geq 1} \{Z_i \geq a - \varepsilon_0\}\right) \leq \frac{E Z_1}{a - \varepsilon_0}$$

From (*)

$$P\left(\sup_{n \geq 1} Z_n \geq a\right) \leq \frac{E Z_1}{a - \varepsilon_0} \quad \forall \ 0 < \varepsilon_0 < a$$

Take $\varepsilon \to 0$ on R.H.S to get

$$\mathbb{P}\left(\sup_{n \geq 1} Z_n \geq a\right) \leq \frac{E Z_1}{a} \qquad \square$$

Theorem 3 :- Let $\{Z_n\}_{n \geq 1}$ be a martingale and assume that

$\exists\ M > 0$ such that $E[Z_n^2] \leq M$, $\forall\ n \geq 1$

Then $\exists$ a r.v. Z such that

$$\mathbb{P}\left(Z_n \longrightarrow Z \text{ as } n \to \infty\right) = 1.$$

Proof:- • $\{Z_n^2\}$ is a sub-martingale.

$\Rightarrow$ (Ex.) $E[Z_n^2 \mid Z_{n-1}, \ldots Z_1] \geq Z_{n-1}^2 \quad \forall\ n \geq 2$

Ex. Take Expectation above

$\Rightarrow E Z_n^2 \geq E Z_{n-1}^2 \quad \forall\ n \geq 2$

$\therefore$ By $E Z_n^2 \leq M \ \forall\ n \geq 1 \Rightarrow \exists\ a \geq 0$ such that

$E Z_n^2 \longrightarrow a$ as $n \to \infty$ — (*)

Hw: let $m \geq 1$ & $Y_n = Z_{n+m} - Z_m \quad \forall n \geq 1$

then $\{Y_n\}_{n \geq 1}$ is a martingale.

Theorem 1 for $|Y_n|$ to get for $b > 0$.

($|Y_n|$ is non-negative submartingale)

$$\mathbb{P}\left(\max_{1 \leq i \leq n} |Y_i| \geq b \right) \leq \frac{E\, Y_n^2}{b^2}$$

$$\Rightarrow \mathbb{P}\left(\max_{1 \leq i \leq n} |Z_{i+m} - Z_m| \geq b \right) \leq \frac{E(Z_{n+m} - Z_m)^2}{b^2}$$

claim 1: $E(Z_{n+m} - Z_m)^2 = E[Z^2_{n+m}] - E[Z_m^2]$

$$\Rightarrow \mathbb{P}\left(\max_{1 \leq i \leq n} |Z_{i+m} - Z_m| \geq b \right) \leq \frac{E[Z^2_{n+m}] - E[Z_m^2]}{b^2}$$

Ex $n \to \infty$ on both sides to get and with (*)

$\forall b > 0$; $\mathbb{P}\left(\bigcup_{i \geq 1} |Z_{i+m} - Z_m| \geq b \right) \leq \frac{\alpha - EZ_m^2}{b^2}$

Ex: Using argument of Theorem 2 proof to get

$\forall b > 0$;

$$0 \leq \mathbb{P}\left(\sup_{i \geq 1} |Z_{i+m} - Z_m| \geq b \right) \leq \frac{\alpha - E Z_m^2}{b^2}$$

$$\Rightarrow \lim_{m \to \infty} \mathbb{P}\left(\sup_{i \geq 1} |Z_{i+m} - Z_m| \geq b \right) = 0 \quad \forall b > 0$$

Claim :- $\mathbb{P}(\{Z_m\}_{m \geq 1}$ is Cauchy sequence $) = 1$

$$\Rightarrow \mathbb{P}(\exists Z : Z_m \to Z \text{ as } n \to \infty) = 1$$

□