Recall:

[Kolmogorov Maximal Inequality]

- $\{Z_n\}_{n\geq 1}$ was a martingale & $\mathbb{E}\, Z_n^2 < \infty$ $\forall n \geq 1$

$$\mathbb{P}\left(\max_{1\leq i\leq n} |Z_i| \geq b\right) \leq \frac{\mathbb{E}\, Z_n^2}{b^2} \qquad \forall n \geq 1,\ \forall b > 0$$

- $\{Z_n\}_{n\geq 1}$ was a non-negative submartingale &

$$\mathbb{P}\left(\max_{1\leq i\leq n} Z_i \geq b\right) \leq \frac{\mathbb{E}\, Z_n}{b} \qquad \forall n \geq 1,\ \forall b > 0$$

[Martingale Convergence Theorem]

Let $\{Z_n\}_{n\geq 1}$ be a martingale and assume that

$\exists\ M > 0$ such that $\mathbb{E}[Z_n^2] \leq M$, $\forall n \geq 1$

Then $\exists$ a r.v. Z such that

$$\mathbb{P}\left(Z_n \longrightarrow Z \text{ as } n \to \infty\right) = 1$$

Proof:

- • $\{Z_n^2\}$ is a sub-martingale.

(Ex.) $\Rightarrow$ $E[Z_n^2 \mid Z_{n-1}, \dots Z_1] \geq Z_{n-1}^2 \quad \forall n \geq 2$

Ex. Take Expectation abov

$\Rightarrow$ $E Z_n^2 \geq E Z_{n-1}^2 \quad \forall n \geq 2$

$\therefore$ By $E Z_n^2 \leq n \quad \forall n \geq 1 \Rightarrow \exists \alpha \geq 0$ Such that

$E Z_n^2 \longrightarrow \alpha$ as $n \to \infty$ — (*)

Hw: let $m \geq 1$ & $Y_n = Z_{n+m} - Z_m \quad \forall n \geq 1$

then $\{Y_n\}_{n \geq 1}$ is a martingale.

By kolmogorov maximal inequality $\forall n \geq 2, \forall b > 0$

$$P\left(\max_{1 \leq i \leq n} |Y_i| \geq b\right) \leq \frac{E Y_n^2}{b^2}$$ — (+)

provided $E Y_n^2 < \infty \quad \forall n \geq 1$

Proof of $EY_n^2 < \infty$:

$$EY_n^2 = E\left[(Z_{n+m} - Z_m)^2\right]$$

$$= E Z_{n+m}^2 + E Z_m^2 - 2 E\left[Z_{n+m} Z_m\right]$$

$$= E Z_{n+m}^2 + E Z_m^2 - 2 E\left[E\left[Z_{n+m} Z_m \mid A_m\right]\right]$$

where A_m is the observable events of $Z_1, \ldots, Z_m$

$E[XY|A_X] = X E[Y|A_X]$

$$= E Z_{n+m}^2 + E Z_m^2 - 2 E\left[Z_m E\left[Z_{n+m} \mid A_m\right]\right]$$

$\{Z_n\}_{n\geq 1}$ is a martingale

$$= E Z_{n+m}^2 + E Z_m^2 - 2 E Z_m^2$$

$$= E Z_{n+m}^2 - E Z_m^2 < \infty.$$

□

By (+)

$$\Rightarrow P\left(\max_{1 \leq i \leq n} |Z_{i+m} - Z_m| \geq b\right) \leq \frac{E(Z_{n+m} - Z_m)^2}{b^2}$$

$$\Rightarrow \quad \mathbb{P}\left(\max_{1 \le i \le n} |Z_{i+m} - Z_m| \ge b \right) \le \frac{\mathbb{E} Z_{n+m}^2 - \mathbb{E} Z_m^2}{b^2}$$

let $n \to \infty$ on both sides

$\mathbb{E} Z_{n+m}^2 \to \alpha$ as $n \to \infty$

$$0 \le \mathbb{P}\left(\bigcup_{i \ge 1} |Z_{i+m} - Z_m| \ge b \right) \le \frac{\alpha - \mathbb{E} Z_m^2}{b^2}$$

Take $m \to \infty$ on both sides ($\mathbb{E} Z_m^2 \to \alpha$ as $m \to \infty$)

$\forall\, b > 0$

$$\lim_{m \to \infty} \mathbb{P}\left(\bigcup_{i \ge 1} |Z_{i+m} - Z_m| \ge b \right) = 0$$

$+ +$

Claim :- $\mathbb{P}(\{Z_m\}_{m \ge 1}$ is Cauchy sequence $) = 1$

Proof :-

$$A = \{ \omega \in \Omega \mid \forall\, b > 0 \ \exists\, N \ge 1 \ |Z_{i+m} - Z_m| < b, \ \forall\, m \ge N \ \& \ i \ge 1 \}$$

To show $\mathbb{P}(A) = 1$.

$$A = \bigcap_{b > 0} \bigcup_{N \ge 1} \bigcap_{m=N}^{\infty} \bigcap_{i=1}^{\infty} \{ |Z_{i+m} - Z_m| < b \}$$

Let $\ell \geq 1$ $\quad b = \frac{1}{2^\ell}$ by (++) $\exists\, k_\ell \geq 1$

$$\mathbb{P}\left(\sup_{i \geq 1} |z_{i+k_\ell} - z_{k_\ell}| > \frac{1}{2^\ell} \right) \leq \frac{1}{2^\ell}$$

$$\Rightarrow \sum_{\ell=1}^{\infty} \mathbb{P}\left(\sup_{i \geq 1} |z_{i+k_\ell} - z_{k_\ell}| > \frac{1}{2^\ell} \right) < \infty$$

By Borel-Cantelli Lemma, if $A_\ell = \left\{ \sup_{i \geq 1} |z_{i+k_\ell} - z_{k_\ell}| > \frac{1}{2^\ell} \right\}$

$$\Rightarrow \mathbb{P}\left(\bigcap_{n \geq 1} \bigcup_{m=n}^{\infty} A_m \right) = 0$$

$$\Rightarrow \mathbb{P}\left(\{ w \in \Omega \mid w \in A_\ell \text{ for infinitely many } \ell \geq 1 \} \right) = 0$$

$$\Rightarrow \mathbb{P}\left(\{ w \in \Omega \mid w \in A_\ell \text{ for finitely many } \ell \geq 1 \} \right) = 1$$

$$\Rightarrow \mathbb{P}\left(\{ w \in \Omega \mid \exists\, \ell_0 \geq 1 \ \forall\, \ell \geq \ell_0, \ \sup_{i \geq 1} |z_{i+k_\ell} - z_{k_\ell}| \leq \frac{1}{2^\ell} \} \right) = 1$$

$\Downarrow$ Ex

$$\Rightarrow \mathbb{P}\left(\bigcap_{b > 0} \bigcup_{N \geq 1} \bigcap_{m=N}^{\infty} \bigcap_{i=1}^{\infty} \{ |z_{i+m} - z_i| < b \} \right) = 1$$

$\Rightarrow \quad \mathbb{P}(\{Z_k\}_{k\geq 1} \text{ is Cauchy sequence}) = 1$

$\Rightarrow \mathbb{P}(\{\omega \in \Omega \mid Z_k(\omega) \longrightarrow Z(\omega) \text{ as } k \to \infty\}) = 1$

Ex. Z is r.v and $\mathbb{P}(Z < \infty) = 1$. □

[Martingale Convergence Theorem 2]

Suppose $\{Z_n\}_{n\geq 1}$ is a martingale and

$$\sup_{n\geq 1} \mathbb{E}|Z_n| < \infty$$ then

$\exists$ a r.v. Z such that

$$\mathbb{P}(Z < \infty) \quad \& \quad \mathbb{P}(Z_n \to Z \text{ as } n \to \infty) = 1.$$

Corollary:

Suppose $\{Z_n\}_{n\geq 1}$ is a non-negative martingale then

$\exists$ a r.v. Z such that

$$\mathbb{P}(Z < \infty) \quad \& \quad \mathbb{P}(Z_n \to Z \text{ as } n \to \infty) = 1.$$

Proof of Corollary: $Z_n \geq 0 \Rightarrow$

$\{Z_n\}_{n\geq 1}$ is a martingale

$$E|Z_n| = EZ_n = EZ_1 < \infty$$

By [Martingale Convergence Theorem 2]

we have the result $\square$

Proof of Martingale Convergence Theorem 2 :

Intuition :-

$x_n \in \mathbb{R}$

$x_n \to x$ as $n \to \infty$

$x_n \to \infty$ as $n \to \infty$

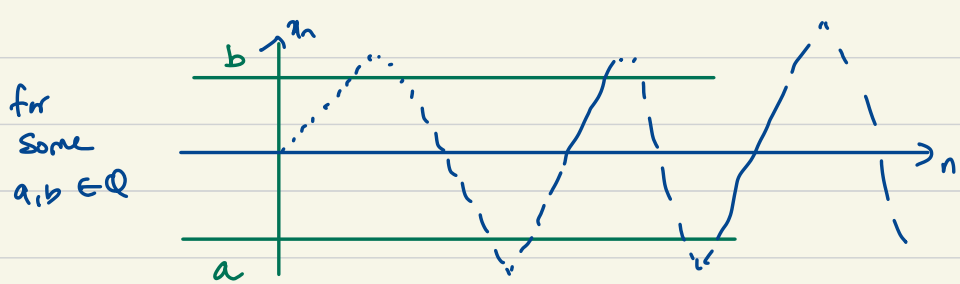
$x_n \to -\infty$ as $n \to \infty$

$$\overline{\lim_{n\to\infty}} x_n = \underline{\lim_{n\to\infty}} x_n$$

x_n does not converge to $a \in \mathbb{R} \cup \{\infty\} \cup \{-\infty\}$

$$\underline{\lim_{n\to\infty}} x_n < \overline{\lim_{n\to\infty}} x_n$$

x_n does not converge to $a \in \mathbb{R} \cup \{\infty\} \cup \{-\infty\}$ $\equiv \bigcup_{a,b \in \mathbb{Q}} \left\{ \underline{\lim_{n\to\infty}} x_n < a < b < \overline{\lim_{n\to\infty}} x_n \right\}$



Crossings of (a,b) is infinite

let $\{Z_n\}_{n\geq 1}$ be a martingale. For $a,b \in \mathbb{Q}$

$$\Lambda_{a,b} = \left\{ \omega \in \Omega \;\middle|\; \underline{\lim_{n\to\infty}} Z_n(\omega) < a < b < \overline{\lim_{n\to\infty}} Z_n(\omega) \right\}$$

$$\mathbb{P}\left(\{Z_n\}_{n\geq 1} \text{ does not converge to } a \in \mathbb{R} \cup \{\pm\infty\} \right)$$

$$= \mathbb{P}\left(\bigcup_{a,b\in\mathbb{Q}} \Lambda_{a,b} \right)$$

$$\underset{Ex}{\leq} \sum_{a,b\in\mathbb{Q}} \mathbb{P}(\Lambda_{a,b})$$

We will first show $\mathbb{P}(\Lambda_{a,b}) = 0 \quad \forall a,b \in \mathbb{Q}$.

let $n \geq 1$ $\quad S_1 = \min\{n \geq 1 \mid Z_n < a\}$

$$t_n = \min\{n \geq S_{n-1} \mid Z_n \geq b\}$$

$n \geq 2$: $\quad S_n = \min\{n \geq t_{n-1} \mid Z_n < a\}$

For $N \geq 1$

$$U_N(a,b) \equiv \max\{k \geq 1 \mid 1 \leq S_1 \leq t_1 \leq \dots < t_k \leq N\}$$

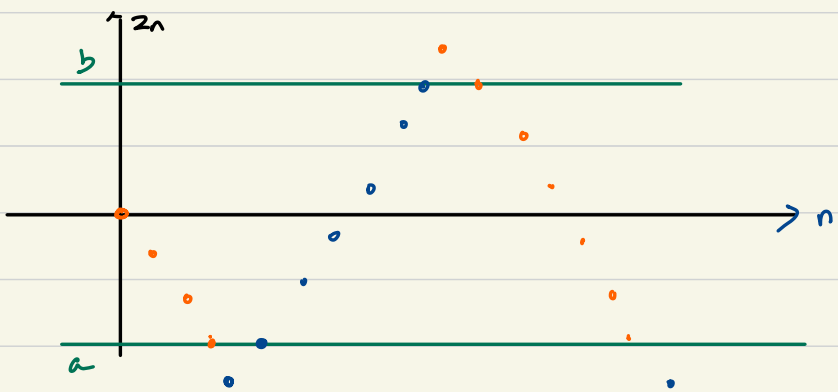
# of crossing of (a,b) until time N.

$N \to \infty \qquad U_N(a,b) \longrightarrow U(a,b)$

$U(a,b) = \#$ of (crossings) of (a,b) by $\{Z_n\}_{n\geq 1}$.

$\therefore \quad \Lambda_{a,b} = \{ U(a,b) = \infty \}.$

E.T.S. $\quad \mathbb{P}(U(a,b) = \infty) = 0$



$V_1 = 1(Z_1 < a)$

$V_n = 1(V_{n-1}=1, Z_{n-1} \leq b) + 1(V_{n-1}=0, Z_{n-1} < a) \quad n \geq 2$

$V_n = \begin{cases} 1 & \text{if below } a \text{ till you reach } b \\ 0 & \text{if above } b \text{ till you reach } a \end{cases}$

Fix $N \geq 1 \qquad Z_N^V = \sum_{k=1}^{N} V_k (Z_k - Z_{k-1})$

Ex: . Z_n^v is a martingale

. $Z_n^v \geq U_n(a,b) \cdot (b-a) - (Z_n - a)^-$

$$\Rightarrow \quad U_n(a,b) \leq \frac{1}{b-a}\left(Z_n^v + (Z_n - a)^- \right)$$

$$\Rightarrow \quad E[U_n(a,b)) \leq \frac{1}{b-a}\left(E Z_n^v + E(Z_n - a)^- \right)$$

$\{Z_n\}_{n\geq 1}$ martingale and $(Z_n - a)^- < Z_n + |a|$

$$\leq \frac{1}{b-a}\left(E Z_1 + E Z_1 + |a| \right)$$

$$\leq C \qquad (\text{say}) \qquad (\ddagger)$$

. $0 \leq U_n(a,b) \uparrow U(a,b)$ as $n \to \infty$

(Ex). $E(U_n(a,b)) \uparrow E\, U(a,b)$

Using this and $(\ddagger)$ we have

$$E\, U(a,b) \leq C$$

$$\Rightarrow \quad \mathbb{P}(U(a,b) < \infty) = 1$$

$$\Rightarrow \quad \mathbb{P}(U(a,b) = \infty) = 0$$

$\Rightarrow \quad P(\ Z_n \to Z \text{ as } n \to \infty \quad Z \in \mathbb{R} \cup \{\infty\} \cup \{-\infty\}) = 1$

(Ex) Z is r.v

Consider $E|Z| = E \lim_{n\to\infty} |Z_n|$

$\overset{\text{Fatou's lemma}}{\leq} \lim_{n\to\infty} E|Z_n|$

Assumption

$\leq \sup_{n \geq 1} E|Z_n| < \infty$

$\Rightarrow \quad P(|Z| < \infty) = 1$

$\Rightarrow \quad P(\ Z_n \to Z \text{ as } n \to \infty \quad Z \in \mathbb{R}) = 1$ □

Walds Identities

Theorem 1: Let $\{X_n\}_{n\geq 1}$ be i.i.d. X_i with $E[X] < \infty$

Let $S_n = X_1 + \dots X_n \quad \forall n \geq 1$

If J is a stopping time & $E[J] < \infty$

Then

$$E\, S_J = E[X]\, E[J]$$

Proof (Sketch): "$S_J = \sum_{n=1}^{J} X_n$"

$$\overset{(Ex.)}{=} \sum_{n=1}^{\infty} X_n \, 1_{J \geq n}$$

$$\overset{(Ex)}{E[S_J]} = \sum_{n=1}^{\infty} E[X_n \, 1_{J \geq n}]$$

$1_{J \geq n}$ - observable from $X_1, X_2, \ldots, X_{n-1}$

$$\overset{(Ex)}{=} \sum_{n=1}^{\infty} E[X_n] \, P(J \geq n)$$

$$\overset{Ex}{=} E[X] \sum_{n=1}^{\infty} P(J \geq n)$$

$$= E[X] \, E[J] \qquad \square$$

let $\{X_n\}_{n \geq 1}$ be i.i.d. X such that

$\exists \, \delta > 0 \quad g(s) = E[e^{sX}] < \infty \quad s \in (-\delta, \delta)$

Let $K(s) = \log g(s)$ & $s \in (-\delta, \delta)$

$S_n = X_1 + X_2 \ldots + X_n$

$$Z_n = \exp(r S_n - n k(r))$$

$$= \prod_{i=1}^{n} \exp(r X_i - k(r))$$

let $Y_i = \exp(r X_i - k(r))$ $\forall i \geq 1$ & independent

$E Y_i = 1$ by definition of $k(r)$

$\Rightarrow \{Z_n\}_{n \geq 1}$ is a product martingale

Theorem 2 If J is a stopping time for $\{Z_n\}_{n \geq 1}$

(w.r.t $\mathcal{A}_n$ – observable by $X_1, \dots, X_n$)

If $E|Z_J| < \infty$ and $\lim_{n \to \infty} E(Z_n | J \geq n) P(J \geq n) = 0$

Then

$$E\left[e^{r S_J - J k(r)}\right] = 1$$

Proof: Apply OST to $Z_n = e^{r S_n - n k(r)}$ $\forall n \geq 1$.

□

Example:

Let $\{X_n\}_{n\geq 1}$ be i.i.d. X such that $X \neq 0$

$\exists\ \delta > 0 \qquad g(s) = E[e^{sX}] < \infty \qquad s \in (-\delta, \delta)$

$S_n = X_1 + X_2 \cdots + X_n$

$\beta, \alpha \in \mathbb{Z}$

$\beta < 0 < \alpha \ ; \ J = \min\{n \geq 1 \mid S_n \geq \alpha \text{ or } S_n \leq \beta\}$

Q1:- Under what assumption on X is

$P(J < \infty) = 1$ & $E[J^n] < \infty$.

A1:- Assume $P(X \in \{-1, 1\}) = 1$ — (I)

- $X \neq 0$, $\beta, \alpha \in \mathbb{Z}$ $\beta < 0 < \alpha$

$\Rightarrow \exists\ n_0 \in \mathbb{N}$ $P(S_{n_0} < \beta - \alpha) > 0$ or

$P(S_{n_0} > \alpha - \beta) > 0$

Set $\varepsilon = \max\{P(S_{n_0} < \beta - \alpha), P(S_{n_0} > \alpha - \beta)\}$

$\Rightarrow$ (Ex.) $P(J \geq n_0) < 1 - \varepsilon$.

(Ex.) $\Rightarrow \quad \forall k \geq 1, \quad \mathbb{P}\left(J \geq k n_0 \,\middle|\, J \geq n_0 (k-1) \right) < 1-\varepsilon$

involves $S_{n_0(k-1)+1}, \dots, S_{n_0 k}$

Ex $\Rightarrow$ $\quad \mathbb{P}(J \geq k n_0) = \prod_{s=1}^{k} \mathbb{P}\left(J \geq n_0 s \mid J \geq n_0 (s-1) \right)$

$$< (1-\varepsilon)^k \qquad \forall k \geq 1 \quad -(*)$$

Ex $\Rightarrow$ $\quad \mathbb{P}(J < \infty) = 1$

and

$$\forall n \geq 1; \quad \mathbb{E}[J^n] = \sum_{k=1}^{\infty} n k^{n-1} \mathbb{P}(J \geq k)$$

As $\quad \mathbb{P}(J \geq k) \leq (1-\varepsilon)^{k/n_0}$

Ex: $\quad \exists C > 0, \ p > 1 \quad$ s.t. : $\quad n k^{n-1} \mathbb{P}(J \geq k) \leq \dfrac{C}{k^p} \qquad \forall k \geq 1$

$$\Rightarrow \quad \mathbb{E}[J^n] < \infty.$$

□

Q2: Under what assumption on X is (OST) hypothesis is satisfied.

A2: Assume (I)

Ex $\Rightarrow \quad S_J \in \{\alpha, \beta\}$

$\Rightarrow \quad |S_J| \leq |\alpha| + |\beta|$

$\Rightarrow \quad |Z_J| \leq e^{\gamma(|\alpha| + |\beta|)}$

$\Rightarrow \quad E|Z_J| < \infty.$ — (*)

Also

$$E(Z_J \mid J > n)\, P(J \geq n) \leq e^{\gamma(|\alpha| + |\beta|)}\, P(J \geq n)$$

By Q1 & A1

as $P(J \geq n) \longrightarrow 0$ as $n \to \infty$

$\therefore \lim_{n \to \infty} E(Z_J \mid J > n)\, P(J \geq n) = 0$ — (*')

$\therefore$ (OST) hypothesis is satisfied. □

Ex (*) (*') also is true if Range $(X) < \infty$.

□