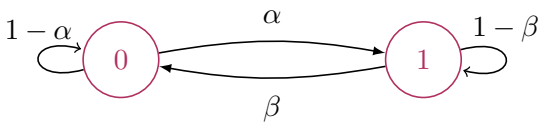
In the next example we consider a two state Markov chain where we shall explicitly compute the distribution of X_n and its limit.

EXAMPLE 1.1.4. Let $0 < \alpha, \beta < 1$. Consider a Markov chain X_n on state space $S = \{0, 1\}$,with $P(X_0 = 0) = \mu_0 \in (0, 1), P(X_0 = 1) = \mu_1 \in (0, 1)$, and transition matrix

$$\mathbf{P} = \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix}.$$

The above dynamics of the chain can be understood via a directed graph with S as the vertex set.



Suppose we wish to determine the n-step transition probabilities $p_{ij}^{[n]}$. From the Chapman-Kolmogorv equations applied to the relation $\mathbf{P}^n = \mathbf{P}^{n-1}\mathbf{P}$, we obtain

$$\begin{aligned} p_{00}^{[n]} &= p_{00}^{[n-1]}(1-\alpha) + p_{01}^{[n-1]}\beta \\ &= p_{00}^{[n-1]}(1-\alpha-\beta) + \beta, \end{aligned}$$

w as $p_{00}^{[n-1]} + p_{01}^{[n-1]} = 1$. This formula provides a recursive method for determining $p_{00}^{[n]}$ from $p_{00}^{[n-1]}$ and there is a standard technique to solve such recursions explicitly. As, $\mathbf{P}^0 = I$ we have $p_{00}^{[0]} = 1$. First we check if there is a constant solution, say $p_{00}^{[n]} = p$. Then p would have to satisfy,

$$p = p(1-\alpha-\beta) + \beta \qquad \Longrightarrow \qquad p = \frac{\alpha}{\alpha+\beta}$$

Set $x_n = p_{00}^{[n]} - \frac{\beta}{\alpha+\beta}$. So $x_0 = 1 - \frac{\beta}{\alpha+\beta} = \frac{\alpha}{\alpha+\beta}$, and x_n can be expressed as

$$\begin{aligned} x_n &= p_{00}^{[n]} - \frac{\beta}{\alpha+\beta} \\ &= p_{00}^{[n-1]}(1-\alpha-\beta) + \beta - \frac{\beta}{\alpha+\beta} \\ &= x_{n-1}(1-\alpha-\beta). \end{aligned}$$

This implies that

$$x_n = (1-\alpha-\beta)^n x_0 = \frac{\alpha}{\alpha+\beta}(1-\alpha-\beta)^n.$$

Therefore

$$p_{00}^{[n]} = \frac{\alpha}{\alpha+\beta}(1-\alpha-\beta)^n + \frac{\beta}{\alpha+\beta}. \tag{1.1.10}$$

Imitating the above we can also obtain,

$$p_{10}^{[n]} = \frac{\beta}{\alpha+\beta} - \frac{\beta}{\alpha+\beta}(1-\alpha-\beta)^n. \tag{1.1.11}$$

With these transition probabilities we can then determine how likely it is that the chain is in any given state at time n.

Note,

$$\begin{aligned} P(X_n = 0) &= P(X_n = 0, X_0 = 0) + P(X_n = 0, X_0 = 1) \\ &= P(X_n = 0 \mid X_0 = 0)P(X_0 = 0) + P(X_n = 0 \mid X_0 = 1)P(X_0 = 1) \\ &= p_{00}^{[n]}\mu(\{0\}) + p_{10}^{[n]}\mu(\{1\}) \\ &= (p_{00}^{[n]} - p_{10}^{[n]})\mu(\{0\}) + p_{10}^{[n]}, \end{aligned} \tag{1.1.12}$$

where we used $\mu(\{0\}) + \mu(\{1\}) = 1$.

Substituting (1.1.10) and (1.1.11) into (1.1.12) we have

$$\mathbb{P}(X_n = 0) = \frac{\beta}{\alpha+\beta} + (1-\alpha-\beta)^n(\mu(\{0\}) - \frac{\beta}{\alpha+\beta}), \tag{1.1.13}$$

and using $\mathbb{P}(X_n = 1) = 1 - \mathbb{P}(X_n = 0)$ we have

$$\mathbb{P}(X_n = 1) = \frac{\alpha}{\alpha+\beta} + (1-\alpha-\beta)^n(\mu(\{1\}) - \frac{\alpha}{\alpha+\beta}), \tag{1.1.14}$$

With a clear explicit expression for the probabilities two observations are immediate:

(a) As $\mid 1-\alpha-\beta \mid < 1$ we have,

$$\lim_{n\to\infty} \mathbb{P}(X_n = 0) = \frac{\beta}{\alpha+\beta} \text{ and } \lim_{n\to\infty} \mathbb{P}(X_n = 1) = \frac{\alpha}{\alpha+\beta}.$$

Thus regardless of the initial distribution μ the distribution of the chain on $\{0,1\}$ a $(\frac{\beta}{\alpha+\beta}, \frac{\alpha}{\alpha+\beta})$. This is called the *limiting distribution.*

(b) Secondly, suppose $\mu(\{0\}) = \frac{\beta}{\alpha+\beta}$ then $\mu(\{1\}) = \frac{\alpha}{\alpha+\beta}$ and this forces,

$$\mathbb{P}(X_n = 0) = \frac{\beta}{\alpha+\beta} \text{ and } \mathbb{P}(X_n = 1) = \frac{\alpha}{\alpha+\beta}$$

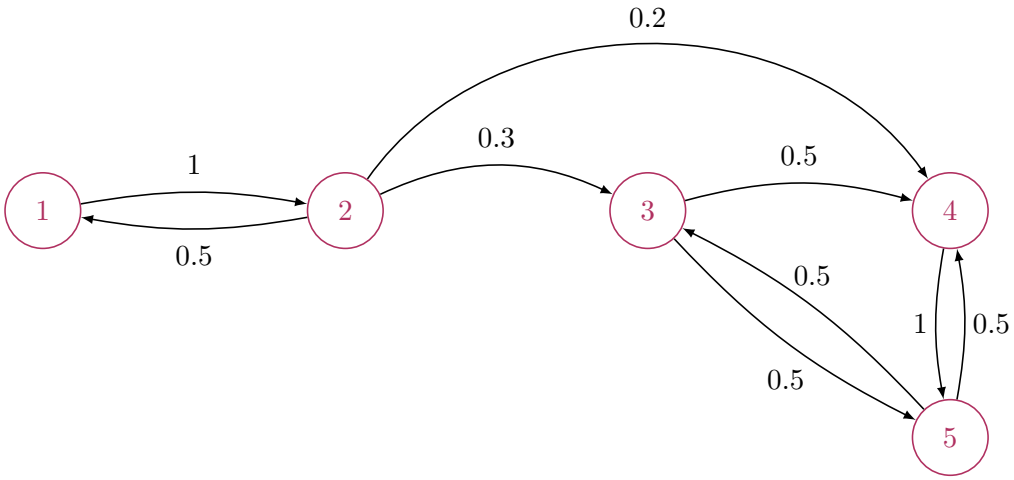
for all $n \geq 1$. Thus if you start with this particular initial distribution then the law of X_n does not change for any $n \geq 1$. Such a distribution is called *stationary distribution.* We shall return to this in Section 1.3.

1.2 COMMUNICATING CLASSES

EXAMPLE 1.2.1. Consider a Markov chain X_n on state space $S = \{1, 2, 3, 4, 5\}$, $X_0 \sim$ Uniform $\{1, 2, 3, 4, 5\}$, and transition matrix

$$\mathbf{P} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0.5 & 0 & 0.3 & 0.2 & 0 \\ 0 & 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0.5 & 0.5 & 0 \end{bmatrix}.$$

The above dynamics of the chain can be understood via a directed graph with S as the vertex set.



From the graph we can make the following elementary observations. Once the chain reaches a state in the set $\{3, 4, 5\}$ it remains within that set of states. (The states $\{1, 2\}$ are not accessible after this point in time). Intuitively, the fraction of time the chain spends in the states $\{1, 2\}$ should tend to zero as time goes to infinity. This does not seem to be the case with $\{3, 4, 5\}$. In fact, one can show that the fraction of time spent at 5 is twice that of 3 which is twice that of 4. Alternatively, using Linear Algebra, one could try to show that

$$\lim_{n\to\infty} p_{11}^{[n]} = 0, \qquad \lim_{n\to\infty} p_{12}^{[n]} = 0, \qquad \lim_{n\to\infty} p_{13}^{[n]} = \frac{2}{7}, \qquad \lim_{n\to\infty} p_{14}^{[n]} = \frac{1}{7}, \qquad \lim_{n\to\infty} p_{15}^{[n]} = \frac{4}{7}.$$

In Example 1.2.1, if the chain ever reaches one of the states 3, 4, or 5, then it may never again visit states 1 or 2. Hence some states in S are not always (i.e. at all times) accessible by the chain with positive probability. In this section we shall identify and classify states based on the extent of their accessibility from other states.

DEFINITION 1.2.2. *Let X_n be a Markov Chain on S with transition matrix $\mathbf{P}$ and initial distribution μ.*

(a) A state $j \in S$ is said to be accessible from another state $i \in S$ if there is an $n \geq 0$ such that $p_{ij}^{[n]} > 0$. This is denoted by $j \leftarrow i$.

(b) Two states i and j in S are said to communicate if i and j are each accessible from the other. This is denoted by $j \leftrightarrow i$.

(c) A subset $C \subset S$ is said to be a communicating class if $i \leftrightarrow j$ whenever $i, j \in C$. Such a set is further called a closed communicating class if $k \not\leftarrow j$ whenever $j \in C$ and $k \in S \backslash C$.

(d) X_n is said to be irreducible if $i \leftrightarrow j$ for all $i, j \in S$.

(e) The period of $i \in S$ is defined as

$$d(i) = g.c.d.\{m \geq 1 : p_{ii}^{[m]} > 0\} \tag{1.2.1}$$

If $p_{ii}^{[m]} = 0$ for all $m \geq 1$, then $d(i)$ is defined to be 0. Finally, X_n is said to be aperiodic if $d(i) = 1$ for all $i \in S$.

In Example 1.2.1, $2 \leftrightarrow 1$, $3 \leftarrow 2$ (since $p_{23} > 0$), $4 \leftarrow 2$ (since $p_{24} > 0$), and $5 \leftarrow 2$ (since $p_{25}^{[2]} > 0$). Moreover $\{1, 2\}$ is a communicating class and $\{3, 4, 5\}$ is a closed communicating class for the Markov Chain. However, the chain is not irreducible since, for example, $2 \not\leftarrow 3$.

LEMMA 1.2.3. *Let X_n be a Markov chain on S with transition matrix $\mathbf{P}$ and initial distribution μ. For $i, j \in S$, $i \neq j$, $j \leftarrow i$ if and only if there exists $n \geq 1$ and $i_k \in S$ for $1 \leq k \leq n-1$ such that*

$$p_{ii_1} p_{i_1 i_2} \cdots p_{i_{n-1} j} > 0$$

Proof: By definition, $j \leftarrow j$ if and only if there exists $n \geq 1$ such that $p_{ij}^{[n]} > 0$. Further, by matrix multiplication rules,

$$p_{ij}^{[n]} = \sum_{i_1, i_2, \ldots i_{n-1} \in S} p_{ii_1} p_{i_1 i_2} \cdots p_{i_{n-1} j}.$$

Therefore $p_{ij}^{[n]} > 0$ if and only if there exists $i_k \in S$ for $1 \leq k \leq n-1$ such that

$$p_{ii_1}p_{i_1i_2}\dots p_{i_{n-1}j} > 0.$$

The result then immediately follows. ■

LEMMA 1.2.4. *$\leftrightarrow$ defined above is an equivalence relation on S. That is, the $\leftrightarrow$ relation satisfies three particular properties:*

(a) (Reflexive property) For all $i \in S$ $\quad i \leftrightarrow i$;

(b) (Symmetric property) For all $i, j \in S$ if $i \leftrightarrow j$ then $j \leftrightarrow i$; and

(c) (Transitive property) For all $i, j, k \in S$ if $i \leftrightarrow j$, $j \leftrightarrow k$ then $i \leftrightarrow k$.

Proof: (a) As $p_{ii}^{[0]} = 1$, then $i \leftrightarrow i$ for all $i \in S$.

(b) This is clear from definition if $\leftrightarrow$. (c) Suppose there is a $i, j, k \in S$ such that $i \leftrightarrow j$, $j \leftrightarrow k$ then $i \leftrightarrow k$. Therefore there are $r, s, m, n \geq 0$ such that $p_{ij}^{[r]} > 0$, $p_{ji}^{[s]} > 0$, $p_{jk}^{[m]} > 0$ and $p_{kj}^{[n]} > 0$. Using (1.1.8), we then have that

$$p_{ik}^{[r+m]} \geq p_{ij}^{[r]}p_{jk}^{[m]} > 0 \text{ and } p_{ki}^{[n+s]} \geq p_{kj}^{[n]}p_{ji}^{[s]} > 0.$$

The above immediately implies $i \leftrightarrow k$. ■

As a consequence of this lemma, the state space S of any Markov chain is always decomposed into disjoint communicating classes. For instance, in Example 1.2.1, the state space $S = \{1, 2\} \cup \{3, 4, 5\}$ is partitioned into two disjoint communicating classes. If S consists a single communicating class then it is a closed communicating class and the Markov chain is irreducible.

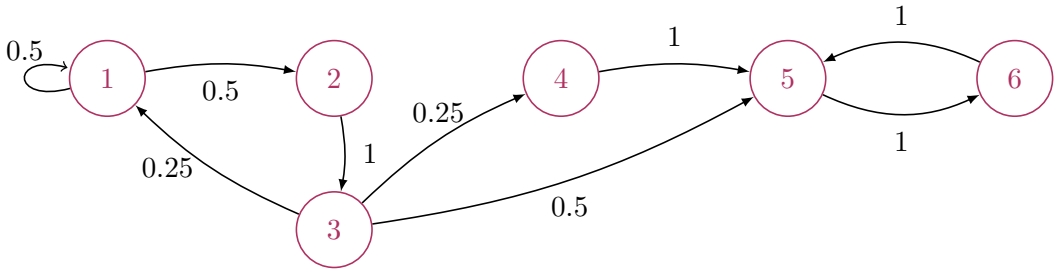
LEMMA 1.2.5. *Let X_n be a Markov chain on S with transition matrix $\mathbf{P}$ and initial distribution μ. For $i, j \in S$, if $i \leftrightarrow j$ then $d(i) = d(j)$.*

Proof: Exercise. ■

EXAMPLE 1.2.6. Consider a Markov chain X_n on state space $S = \{1, 2, 3, 4, 5, 6\}$, and transition matrix

$$\mathbf{P} = \begin{bmatrix} 0.5 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0.25 & 0 & 0 & 0.25 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

The above dynamics of the chain can be understood via a directed graph with S as the vertex set.



First note that states $1, 2$, and 3 all communicate with each other. None of these states is accessible from any others, though, so $\{1, 2, 3\}$ is a communicating class. However, this is not a closed communicating class because other states (such as state 4) are accessible from this class. Usisng similar reasoning $\{4\}$ is a communicating class of its own. No other state is in this class because state 4 doesn't communicate with any other state. Finally, $\{5, 6\}$ is a closed communicating class because 5 and 6 communicate with each other and no other states are accessible from either 5 or 6.

We may also calculate the period of each state.

- Note $p_{11} > 0$ so $d(1) = 1$.
- Next note that $p_{22}^{[3]} > 0$ and $p_{22}^{[4]} > 0$ so $d(2) = 1$. Similarly $d(3) = 1$ since $p_{33}^{[3]} > 0$ and $p_{33}^{[4]} > 0$.
- Because a Markov chain in state 4 must leave immediatley and never return (i.e. $p_{44}^{[n]} = 0$ for every $n \geq 1$) we have $d(4) = 0$.
- Finally, note that a Markov chain in state 5 is guaranteed to be in state 6 in one step and return to state 5 again in the following step. So it is easy to see that $p_{55}^{[n]} > 0$ exactly when n is even. Therefore $d(5) = g.c.d.\{2, 4, 6, \dots\} = 2$. Similiar reasonining shows $d(6) = 2$.

It is worth noting that in this example, all states in a given communicating class share the same period. In fact, this must always happen – a fact you are asked to prove in the exercises below. The period of each of the elements in a communicating class is referred to as the class's periodicity. ■

1.3 STATIONARY DISTRIBUTION

In Example 1.1.4, we calculated the limiting distribution from the explicit formuale available (1.1.13) and (1.1.14). It also had the property that, if it were selected as the

initial distribution, then the law of X_n does not change for any $n \geq 1$. Thus we were able to identify a stationary distribution. One could also ask alternatively, if we are given a probability measure π on S, when will be it a stationary measure. We would require that $X_n \sim \pi$ for all $n \geq 0$. Then applying (1.1.2) for $n = 1$ we would conclude that for $j \in S$,

$$\pi(j) = P(X_1 = j) = \sum_{j \in S} p_{ij} P(X_0 = i) = \sum_{j \in S} p_{ij} \pi(i).$$

Indeed this is enough to characterise a stationary stationary distribution. Thus we arrive at the below definition.

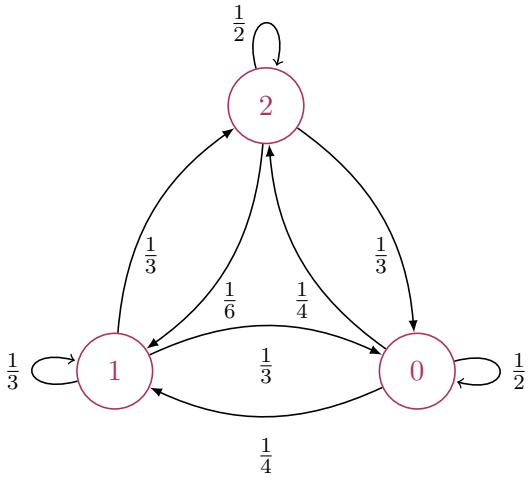
DEFINITION 1.3.1. *A probability measure π on S is said to be a* **stationary distribution** *for the Markov chain X_n or its if*

$$\sum_{i \in S} \pi(i) p_{ij} = \pi(j). \tag{1.3.1}$$

EXAMPLE 1.3.2. Consider a Markov chain X_n on state space $S = \{0, 1, 2\}$ and transition matrix

$$\mathbf{P} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{bmatrix}.$$

The above dynamics of the chain can be understood via a directed graph with S as the vertex set.



If a stationary distribution π exists, it must satisfy:

$$\frac{\pi(\{0\})}{3} + \frac{\pi(\{1\})}{4} + \frac{\pi(\{2\})}{6} = \pi(\{0\}) \tag{1.3.2}$$
$$\frac{\pi(\{0\})}{3} + \frac{\pi(\{1\})}{2} + \frac{\pi(\{2\})}{3} = \pi(\{1\}) \tag{1.3.3}$$
$$\frac{\pi(\{0\})}{3} + \frac{\pi(\{1\})}{4} + \frac{\pi(\{2\})}{2} = \pi(\{2\}) \tag{1.3.4}$$
$$\pi(\{0\}) + \pi(\{1\}) + \pi(\{2\}) = 1 \tag{1.3.5}$$

We leave it to the reader to verify this system of equations has a unique solution:

$$\pi(\{0\}) = \frac{6}{25}, \pi(\{1\}) = \frac{10}{25}, \pi(\{2\}) = \frac{9}{25}$$

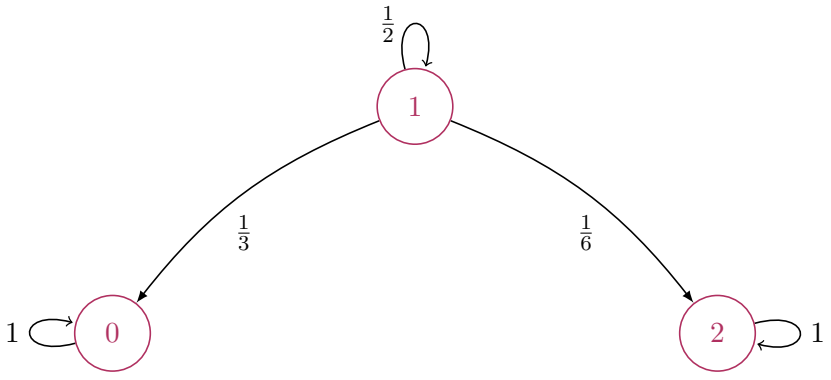
so this is the unique stationary distribution for X_n. ■

We conclude this section by understanding connections between communicating classes and stationary distributions. It is not necessary that there is a unique stationary distribution for a Markov chain. We shall illustrate this when there are distinct communicating classes.

EXAMPLE 1.3.3. Let X_n be a Markov chain on state space $S = \{0, 1, 2\}$ and transition matrix

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \\ 0 & 0 & 1 \end{bmatrix}.$$

which may be represented the graph



As $p_{ii} > 0$ for $i = 1, 2, 3$, $p_{10} > 0 p_{12} > 0$ we conclude that S can be thought of three communicating classes $\{1\}$ and the other two $\{2\}$, $\{0\}$ being closed communicating classes.

Any stationary distribution of this Markov chain must satisfy:

$$\pi(\{0\}) + \frac{\pi(\{1\})}{3} = \pi(\{0\}) \tag{1.3.6}$$

$$\frac{\pi(\{1\})}{2} = \pi(\{1\}) \tag{1.3.7}$$

$$\frac{\pi(\{1\})}{6} + \pi(\{2\}) = \pi(\{2\}) \tag{1.3.8}$$

$$\pi(\{0\}) + \pi(\{1\}) + \pi(\{2\}) = 1 \tag{1.3.9}$$

The second equation in the system makes it apparent that $\pi(\{1\}) = 0$, but it follows that $\pi(\{0\})$ and $\pi(\{2\})$ are arbitrary provided they sum to 1. Therefore any distribution of the form

$$\pi(\{0\}) = p, \pi(\{1\}) = 0, \pi(\{2\}) = 1 - p$$

where $0 \leq p \leq 1$ will be a stationary distribution for X_n. ■

1.4 HITTING PROBABILITIES AND EXPECTED HITTING TIMES

In this section we will analyze two problems which may be solved using similar techniques. The first problem is determining how likely it is that a Markov chain will visit one state before another (or whether it will ever visit a specified state at all). The second problem is determining how long it takes before a Markov chain reaches a certain state. We begin with some notation. Let X_n be a Markov chain on S with transition matrix $\mathbf{P}$ and initial distribution μ. If $i \in S$, we shall let P_i denote the probability P corresponding to the initial distribution $\mu(\{i\}) = 1$. Thus when distributed according to P_i the Markov chain starts in the state i, i.e $\mu(\{i\}) = 1 = P_i(X_0 = i)$.

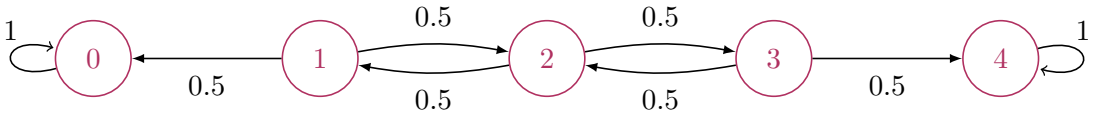
Let us consider the following example.

EXAMPLE 1.4.1. Suppose Laxmi and Maria have two rupees each. They decide to play a game according to the following rules. At each turn a coin will be tossed, if it turns up heads then Laxmi will give Maria one rupee. If it turns up tails then Maria will give Laxmi one rupee. The game ends if any one player runs out of money. Let X_n be the wealth

of, say, Laxmi at time n. It can be seen easily that X_n is a Markov chain on state space $S = \{0, 1, 2, 3, 4\}$ with $P(X_0 = 2) = 1$ and transition matrix

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The above dynamics of the chain can be understood via a directed graph with S as the vertex set.



It is clear from above that $\{0\}$ and $\{4\}$ are closed communicating classes. Once the chain hits 0 (Maria wins the game) or the chain hits 4 (Laxmi wins the game) the chain remains there forever. Such states are called absorbing states. One natural question of interest is who wins the game. Another is how long the game will continue until there is a winner. Both questions may be reformulated in terms of the Markov chain. If the Markov chains visits 0 before visting 4 then Maria wins the game, but if 4 is visited first, then Laxmi wins. The value of n at the time the chain reaches either of these states is a random quantity, but it represents the game's duration. The answer to such questions requires analysis of the times the chain first visits certain sites. These are called hitting times and we will give precise definitions for this concept below. ■

DEFINITION 1.4.2. *Let X_n be a Markov chain on S with transition matrix $\mathbf{P}$ and initial distribution μ.*

(a) A state $j \in S$ is said to be absorbing if $\{j\}$ is a closed communicating class.

(b) Let $A \subset S$ and define the hitting time *of A as*

$$T^A = \min\{k \geq 0 : X_k \in A\}.$$

We will use the convention that $\min\{\emptyset\} = \infty$, so either T^A is a non-negative integer equal to the first time the chain visits the set A or $T^A = \infty$ in which case no state in the set A is ever reached. Note that since T^A depends on the path of the Markov chain it is inherantly a random variable.

EXAMPLE 1.4.3. (Example 1.4.1 Contd.) Let $A = \{4\}$. As we saw earlier 4 is an absorbing state. If the chain visits 4 then Laxmi wins the game. So the probability that Laxmi wins with initial capital 2 can be reformulated as

$$\mathbb{P}(X_n = 4 \text{ for some } n \geq 0 \mid X_0 = 2) = \mathbb{P}(T^A < \infty \mid X_0 = 2).$$

Finding the above probability involves a technique that uses a one-step decomposition of the chain to obtain a recursion relation. In this example we will first intuitively derive the recursion relation and also solve it.

Define a function $h : S \to [0, 1]$ to be given by $h(i) = \mathbb{P}(T^A < \infty \mid X_0 = i)$. This is the probability the chain ever reaches state 4 if it starts in state i, so we are interested in finding $h(2)$. First we observe

$$h(4) = \mathbb{P}(T^A < \infty, \mid X_0 = 4) \geq \mathbb{P}(T^A = 0 \mid X_0 = 4) = \mathbb{P}(X_0 = 4 \mid X_0 = 4) = 1.$$

As this quantity is a probability we also know $h(4) \leq 1$ which implies that

$$h(4) = 1. \tag{1.4.1}$$

Second we observe for all $n \geq 1$, $\mathbb{P}(X_n = 0 \mid X_0 = 0) = 1$ implying $P(X_n = 4 \text{ for some } n \geq 0 \mid X_0 = 0) = 0$. Therefore,

$$h(0) = 0. \tag{1.4.2}$$

Let us consider $h(2)$. Here the chain starts at 2 and wemust find the probability that it will reach 4. In one step the chain can either move to 1 or to 3 each with probability $\frac{1}{2}$. Once the chain reaches 1 or 3 then by Markov property it is enough to calculate the probability of hitting 4 starting at 1 or 3. Thus by conditioning on the next step and using the Markov property we have

$$\begin{aligned}
h(2) &= \mathbb{P}(X_n = 4 \text{ for some } n \geq 0 | X_0 = 2) \\
&= \mathbb{P}(X_n = 4 \text{ for some } n \geq 0 | X_0 = 2, X_1 = 1)\mathbb{P}(X_1 = 1 | X_0 = 2) \\
&\quad + \mathbb{P}(X_n = 4 \text{ for some } n \geq 0 | X_0 = 2, X_1 = 3)\mathbb{P}(X_1 = 3 | X_0 = 2) \\
&= \mathbb{P}(X_n = 4 \text{ for some } n \geq 0 | X_0 = 1)(\frac{1}{2}) + \mathbb{P}(X_n = 4 \text{ for some } n \geq 0 | X_0 = 3)(\frac{1}{2}) \\
&= \frac{1}{2}h(1) + \frac{1}{2}h(3).
\end{aligned}$$

We do not know the values of $h(1)$ or $h(3)$ but doing the exactly similar conditioning on the next step and using the Markov Property we can see that

$$h(3) = \frac{1}{2}h(2) + \frac{1}{2}h(4) \tag{1.4.3}$$

$$h(1) = \frac{1}{2}h(2) + \frac{1}{2}h(0) \tag{1.4.4}$$

Using (1.4.1), (1.4.2), (1.4.3), and (1.4.4) in (1.4.3) we have

$$h(2) = \frac{1}{4}h(2) + \frac{1}{4}h(2) + \frac{1}{4}$$

which implies $h(2) = \frac{1}{2}$. So the probability that Laxmi will win the game is $\frac{1}{2}$. ■

The method of conditioning on one step and using the Markov property is widely applicable to caculate hitting distributions of sets in general. We present it in the next proposition.

PROPOSITION 1.4.4. *Let X_n be a Markov chain on S with transition matrix $\mathbf{P}$ and initial distribution μ. Let $A \subset S$ and T^A be the hitting time of the set A. Let $h^A : S \to [0,1]$ be given by*

$$h^A(i) = \mathbb{P}(T^A < \infty \mid X_0 = i).$$

(a) h^A is a solution of the linear system of equations given by

$$h^A(i) = \begin{cases} 1 & \text{if } i \in A \\ \sum_{j \in S} p_{ij} h^A(j) & \text{if } i \notin A \end{cases}$$

(b) If $f : S \to [0, \infty)$ is another solution to (a) then $f(i) \geq h(i)$ for all $i \in S$.