PROPOSITION 1.4.5. *Let X_n be a Markov chain on S with transition matrix $\mathbf{P}$ and initial distribution μ. Let $A \subset S$ and T^A be the hitting time of the set A. Let $h^A : S \to [0,1]$ be given by*

$$h^A(i) = \mathbb{P}(T^A < \infty \mid X_0 = i).$$

(a) h^A is a solution of the linear system of equations given by

$$h^A(i) = \begin{cases} 1 & \text{if } i \in A \\ \sum_{j \in S} p_{ij} h^A(j) & \text{if } i \notin A \end{cases}$$

(b) If $f : S \to [0, \infty)$ is another solution to (a) then $f(i) \geq h(i)$ for all $i \in S$.

Proof: (Not covered in class but is within the scope of the syllabus and is from [Nor97].)
(a) If $i \in A$ then $\mathbb{P}(X_0 \in A \mid X_0 = i) = 1$. Hence,

$$h^A(i) = \mathbb{P}(T^A < \infty \mid X_0 = i) \geq \mathbb{P}(T^A = 0 \mid X_0 = i) = \mathbb{P}(X_0 \in A \mid X_0 = i) = 1$$

and since $h^A(i) \leq 1$ this implies

$$h^A(i) = 1 \tag{1.4.5}$$

For $i \notin A$, the chain will take at least one time step to enter A if it starts at i. Hence $P_i(T^A \geq 1) = 1$ further, as $\mathbf{P}$ is the one step transition matrix

$$\mathbb{P}(T^A = 1 \mid X_0 = i) = \mathbb{P}(X_1 \in A \mid X_0 = i) = \sum_{j \in A} \mathbb{P}(X_1 = j \mid X_0 = i) = \sum_{j \in A} p_{ij}. \tag{1.4.6}$$

For $n > 1$, by definition of T^A we have

$$\begin{aligned} \{T^A = n\} &= \cup_{j=1}^{n-1} \{X_j \notin A\} \cup \{X_n \in A\}. \\ &= \bigcup_{1 \leq j \leq n-1, i_j \notin A \text{ and } i_n \in A} \{X_1 = i_1, X_2 = i_2, \ldots X_{n-1} = i_{n-1}, X_n = i_n\} \end{aligned}$$

So, the above will imply

$$\begin{aligned} \mathbb{P}(T^A = n \mid X_0 = i) &= \sum_{i_1 \notin A} \sum_{i_2 \notin A} \cdots \sum_{i_{n-1} \notin A} \sum_{i_n \in A} \mathbb{P}(X_1 = i_1, X_2 = i_2, \ldots X_{n-1} = i_{n-1}, X_n = i_n \mid X_0 = i) \\ &= \sum_{i_1 \notin A} \sum_{i_2 \notin A} \cdots \sum_{i_{n-1} \notin A} \sum_{i_n \in A} p_{ii_1} \cdot \prod_{k=1}^{n-1} p_{i_k i_{k+1}} \end{aligned} \tag{1.4.7}$$

Observing that (1.4.7) holds for all $i \notin A$ and $n > 1$. A simple relabelling in (1.4.7) will yield

$$\begin{aligned}\mathbb{P}(T^A = n \mid X_0 = i) &= \sum_{j\notin A} p_{ij} \sum_{i_2\notin A} \sum_{i_3\notin A} \cdots \sum_{i_{n-1}\notin A} \sum_{i_n\in A} p_{ji_2} \cdot \prod_{k=2}^{n-1} p_{i_k i_{k+1}} \\ &= \sum_{j\notin A} p_{ij} \sum_{i_1\notin A} \sum_{i_2\notin A} \cdots \sum_{i_{n-2}\notin A} \sum_{i_{n-1}\in A} p_{ji_1} \cdot \prod_{k=1}^{n-2} p_{i_k i_{k+1}} \\ &= \sum_{j\notin A} p_{ij} \mathbb{P}(T^A = n-1 \mid X_0 = j). \end{aligned} \tag{1.4.8}$$

Now, for $i \notin A$

$$\begin{aligned} h^A(i) = \mathbb{P}(T^A < \infty \mid X_0 = i) &= P(\bigcup_{n=1}^{\infty} \{T^A = n\} \mid X_0 = i) = \sum_{n=1}^{\infty} \mathbb{P}(T^A = n \mid X_0 = i) \\ &= \sum_{j\in A} p_{ij} + \sum_{n=2}^{\infty} \sum_{j\notin A} p_{ij} \mathbb{P}(T^A = n-1 \mid X_0 = j) \\ &= \sum_{j\in A} p_{ij} + \sum_{j\notin A} p_{ij} \sum_{n=2}^{\infty} \mathbb{P}(T^A = n-1 \mid X_0 = j) \\ &= \sum_{j\in A} p_{ij} + \sum_{j\notin A} p_{ij} \mathbb{P}(T^A < \infty \mid X_0 = j) \\ &= \sum_{j\in S} p_{ij} \mathbb{P}(T^A < \infty \mid X_0 = j), \end{aligned} \tag{1.4.9}$$

where we used (1.4.5) in the last equality. Hence by (1.4.5) and (1.4.9) we have the result.

(b) Suppose f solves the system described in (a). Then for $i \in A$ we have $f(i) = 1 = h(i)$, and for $i \notin A$ we have

$$\begin{aligned} f(i) &= \sum_{j\in S} p_{ij} f(j) \\ &= \sum_{j\in A} p_{ij} + \sum_{j\notin A} p_{ij} f(j) \end{aligned} \tag{1.4.10}$$

Now by iterating the method from (1.4.10) used to rewrite the function f we have

$$\begin{aligned} f(i) &= \sum_{j\in A} p_{ij} + \sum_{j\notin A} p_{ij}\left(\sum_{k\in A} p_{jk} + \sum_{k\notin A} p_{jk} f(k)\right) \\ &= \sum_{j\in A} p_{ij} + \sum_{j\notin A}\sum_{k\in A} p_{ij}p_{jk} + \sum_{j\notin A}\sum_{k\notin A} p_{ij}p_{jk}f(k) \\ &= P_i(X_1 \in A) + P_i(X_1 \notin A, X_2 \in A) + \sum_{j\notin A}\sum_{k\notin A} p_{ij}p_{jk}f(k) \\ &= P_i(T^A = 1) + P_i(T^A = 2) + \sum_{j\notin A}\sum_{k\notin A} p_{ij}p_{jk}f(k) \end{aligned} \tag{1.4.11}$$

Proceeding inductively, we obtain for any $n \in \mathbb{N}$, $i \notin A$

$$f(i) = \sum_{m=1}^{n} P_i(T^A = m) + \sum_{i_m \in A^c, 1\le m\le n} p_{ii_1} \prod_{m=2}^{n} p_{i_{m-1}i_m} f(i_n). \tag{1.4.12}$$

From the above we have for any $n \in \mathbb{N}$, $i \notin A$

$$f(i) \ge \sum_{m=1}^{n} P_i(T^A = m)$$

and consequently taking $n \to \infty$ in the above we obtain

$$f(i) \ge \sum_{m=1}^{\infty} P_i(T^A = m) = P_i(T^A < \infty) = h^A(i).$$

This concludes the proof. ∎

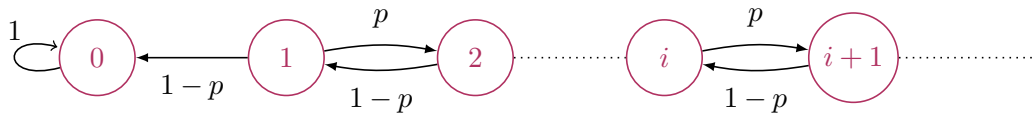
Example 1.4.6. Natasha enters a casino with 10 Rupees. At each time step, a coin is tossed with probability p, $0 < p < 1$ of coming up heads. While she has money, Natasha can play the following game. If the coin comes up heads then the casino pays her one rupee, but if it comes up tails then she pays the casino one rupee. A natural question of interest would be if Natasha will loose all her money to the casino.

Let X_n be the amount of rupees that Natasha has at time n. At any point in time if she has no money left then she leaves the casino. As seen earlier, X_n can be modeled a

Markov chain on state space $S = \{0, 1, \dots\}$ with $P(X_0 = 10) = 1$ and transition matrix $\mathbf{P}$, given by

$$p_{ij} = \begin{cases} 1 & \text{if } i = j = 0 \\ p & \text{if } i \geq 1, j = i + 1 \\ 1 - p & \text{if } i \geq 1, j = i - 1 \\ 0 & \text{otherwise} \end{cases}$$

The dynamics of X_n can be thought of as a Random Walk on $\mathbb{W}$ given by



Clearly $A = \{0\}$ is an absorbing state. Let $h^A : S \to [0, 1]$ to be given by

$$h^A(i) = P(T^A < \infty \mid X_0 = i)$$

and we are interested in $h^A(10)$. Let $q = 1 - p$. From Proposition 1.4.5 we know that

$$h^A(i) = \begin{cases} 1 & \text{if } i = 0 \\ ph^A(i + 1) + qh^A(i - 1) & \text{if } i \geq 1 \end{cases}$$

Solving such a recurrence relation is discussed in Homework 5. From Homework 5, 4(a), if $p \neq q$ then

$$h^A(i) = c_1 + c_2 \left(\frac{q}{p}\right)^i.$$

for some constants c_1 and c_2. As $h^A(0) = 1$ we have $c_1 + c_2 = 1$ which implies

$$h^A(i) = (1 - c_1) \left(\frac{q}{p}\right)^i + c_1.$$

First suppose $q > p$ (which means Natasha is more likely to lose a game than to win it, the usual state of affairs for a casino game). Then c_1 must be zero. Otherwise $h^A(i)$ is unbounded as i increases which is a contradiction since $h^A(i) \leq 1$. Therefore,

$$\text{if } p > q \text{ then } h^A(i) = 1 \text{ for all } i \geq 1.$$

In this case Natasha will loose all her money with probability one if she stays long enough in the casino regardless of how much money she starts with.

Now suppose $q < p$. Since $\left(\frac{q}{p}\right)^i \to 0$ as $i \to \infty$ this will force $c_1 \geq 0$. Since each $c_1 \geq 0$ provides a solution to the recursion and since by Proposition 1.4.5(b) the function $h^A(i)$ is less than or equal to any such solution, this implies that $c_1 = 0$. Therefore

$$\text{if } q < p \text{ then } h^A(i) = \left(\frac{q}{p}\right)^i \text{ for all } i \geq 1.$$

Since Natasha begins with $i = 10$ rupees, there is a $\left(\frac{q}{p}\right)^{10}$ probability that Natasha will loose all her money if she continues betting forever.

Finally suppose $p = q$. Then from Homework 5, 4(b), if $p = q$ then

$$h^A(i) = c_1 + ic_2.$$

Now, $c_1 = 1$ as $h^A(0) = 1$ and $c_2 = 0$ as $0 \leq h^A(i) \leq 1$. This implies

$$h^A(i) = 1 \text{ for all } i \geq 0,$$

and as in $q > p$ case Natasha will loose all her money with probability one if she stays long enough in the casino. Only in the $q < p$ case is there some chance she could continue playing indefinitely. In the case $p = 1$ she surely will. ■

The previous example is known as the Gambler's ruin chain. We shall now discuss a related, but more general model called the Birth-Death chain.

EXAMPLE 1.4.7. Let X_n be a Markov chain on $S = \{0, 1, \dots\}$ with transition matrix $\mathbf{P}$ with

$$p_{ij} = \begin{cases} 1 & \text{if } i = j = 0 \\ p_i & \text{if } i \geq 1, j = i + 1 \\ 1 - p_i & \text{if } i \geq 1, j = i - 1 \\ 0 & \text{otherwise} \end{cases}$$

where we assme $0 < p_i < 1$ for every $i \geq 1$. Note this model is very similar to the Gambler's ruin chain of the previous example except that probability associated with the coin flip is potentially different for every state. Clearly $A = \{0\}$ is an absorbing state. Let $q_i = 1 - p_i$ and let $h^A : S \to [0, 1]$ to be given by

$$h^A(i) = \mathbb{P}(T^A < \infty \mid X_0 = i).$$

From Proposition 1.4.5 we know that

$$h^A(i) = \begin{cases} 1 & \text{if } i = 0 \\ \\ p_i h^A(i+1) + q_i h^A(i-1) & \text{if } i \geq 1 \end{cases}$$

We will not be able to solve the recurrence relation using the techniques of the previous example. Instead, let $u_i = h(i-1) - h(i)$. From the above, we have

$$u_{i+1} = \frac{q_i}{p_i} u_i,$$

for $i \geq 1$. Iterating the above, inductively we obtain for all $i \geq 1$

$$u_{i+1} = \left(\prod_{k=1}^{i} \frac{q_k}{p_k} \right) u_1.$$

On the other hand it is easy to see via a telescoping sum argument that $h^A(0) - h^A(i) = \sum_{j=1}^{i} u_j$, and this implies for $i \geq 1$,

$$h^A(i) = 1 - u_1 \left[\sum_{j=1}^{i} \prod_{k=1}^{j-1} \left(\frac{q_k}{p_k} \right) \right].$$

If $\sum_{j=1}^{\infty} \prod_{k=1}^{j-1} \left(\frac{q_k}{p_k} \right) = \infty$ then as $0 \leq h^A(i) \leq 1$ for all $i \geq 0$ this will force $u_1 = 0$ which will imply $u_i = 0$ for all $i \geq 1$ and consequently

$$h^A(i) = 1 \text{ for all } i \geq 0.$$

If $\sum_{j=1}^{\infty} \prod_{k=1}^{j-1} \left(\frac{q_k}{p_k} \right) < \infty$ then to satisfy the requirement that $0 \leq h^A(i) \leq 1$ it must be that

$$0 \leq u_1 \leq \frac{1}{\sum_{j=1}^{\infty} \prod_{k=1}^{j-1} \left(\frac{q_k}{p_k} \right)}.$$

and so by Proposition 1.4.5(b) this implies that

$$u_1 = \frac{1}{\sum_{j=1}^{\infty} \prod_{k=1}^{j-1} \left(\frac{q_k}{p_k} \right)}.$$

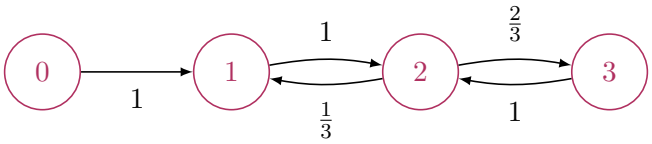
Therefore for $i \geq 1$

$$h^A(i) = \frac{\sum_{j=i+1}^{\infty} \prod_{k=1}^{j-1} \left(\frac{q_k}{p_k} \right)}{\sum_{j=1}^{\infty} \prod_{k=1}^{j-1} \left(\frac{q_k}{p_k} \right)}.$$

1.5 RECURRENCE AND TRANSIENCE

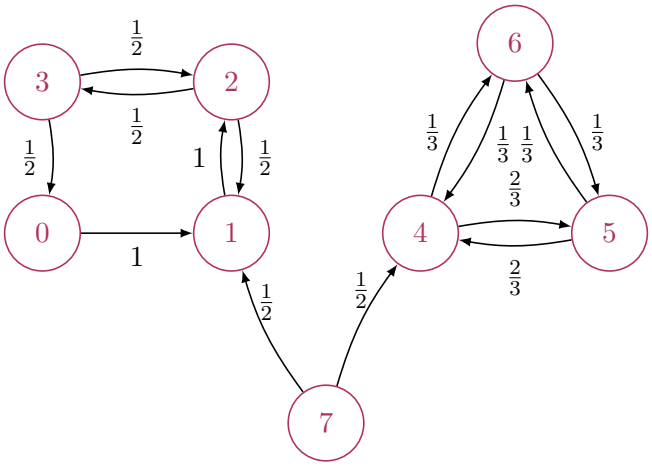
A quantitative way to understand long time behaviour of chain is to decide how many times does it visit a site. We consider a few situations in the next example.

EXAMPLE 1.5.1. First we consider, X_n, to be a Markov chain on $S = \{0, 1, 2, 3\}$ with the dynamics desribed via the graph below.



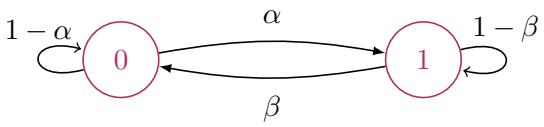
It is immediate that the number of vists to $\{0\}$ (counting from time $n = 0$) is either 1 (when the chain starts at state 0) or0 (since if the chain does not begin at state 0 is can never visit that state in the future. Consequently the set $\{1, 2, 3\}$ will be visited infinitely often by the chain as $n \to \infty$. Though it may not be as apparent, it is a fact that since $1 \leftrightarrow 2 \leftrightarrow 3$, the chain will visit $\{1\}, \{2\}$, and $\{3\}$ infinitely often.

Let us now consider another Markov chain on Y_n on $T = \{0, 1, 2, 3, 4, 5, 6, 7\}$ whose dynamics is given by the graph below.



It is immediate that the number of vists to $\{7\}$ (counting from time $n = 0$) is either 1 (when the chain starts at state 7) or 0 (if the chain stats in any other state). Consequently the sets $\{0, 1, 2, 3\}$ or $\{4, 5, 6\}$ will be visited infinitely often by the chain as $n \to \infty$. Further, a little bit of thought will yield that for some starting points only one of them is visited infintely often and the other set is never visited.

Our final example is the Markov chain from Example 1.1.4 given by dynamics below



If $0 < \alpha, \beta < 1$ then one can show that both 0 and 1 are visited infinitely often by the chain as $n \to \infty$. There are no states in the chain that are visited finitely many times. ■

We shall now formalise the observations from the above example and proceed to establish a dichotomy property for a state in terms of the number of visits to the state by the chain. Let X_n be a Markov chain on S with transition matrix $\mathbf{P}$ and initial distribution μ. . Define number of visits by time n to the state $i \in S$ by

$$N_n(i) = \sum_{m=0}^{n} 1_{\{X_m = i\}} \tag{1.5.1}$$

and the total number of visits to the state $i \in S$ by the chain as

$$N(i) = \sum_{m=0}^{\infty} 1_{\{X_m = i\}}. \tag{1.5.2}$$

A state $i \in S$ is called recurrent if, with probability one, the total number of visits to state i is infinite and is called transient if, with probability one, the total number of visits to state i is finite.

DEFINITION 1.5.2. *A state $i \in S$ is said to be*

1. *recurrent if $\mathbb{P}(N(i) = \infty \mid X_0 = i) = 1$.*
2. *transient if $\mathbb{P}(N(i) < \infty \mid X_0 = i) = 1$.*

The Markov chain X_n is said to be recurrent (resp. transient) if every state is recurrent (resp. transient).

The dichotomy referred to earlier is the fact that a state must be be either recurrent or transient. To establish this fact we will need to introduce the notion of return times. For any $A \subset S$, the return time to A is defined by

$$T_A = \inf\{n \geq 1 : X_n \in A\}. \tag{1.5.3}$$

Note that if $A = \{j\}$ is a single-state set, then $T_{\{j\}} = T_j$ is simply the hitting time of state $\{j\}$.

Suppose a chain starts at state $i \in S$ and visits the i, m times. Then the time length of each of the m excursions between each visit are i.i.d. under P_i by the Markov Property. Thus, the probability that the chain makes at least m visits to a site $i \in S$ is the same as the probabilty that the return time to site i is finite for m independent copies of the chain. This notion is made rigorous in the next lemma where we derive a relationship between $N(i)$ and T_i for $i \in S$.

LEMMA 1.5.3. *Let X_n be a Markov chain on S with transition matrix $\mathbf{P}$ and initial distribution μ. Let $i \in S$. Then for all $m \geq 1$*

$$\mathbb{P}(N(i) \geq m \mid X_0 = i) = [\mathbb{P}(T_i < \infty \mid X_0 = i)]^m \tag{1.5.4}$$

Proof : Let $m \in \mathbb{N}$. First note that

$$\{N(i) \geq m\} = \{N(i) \geq m, T_i < \infty\} = \bigcup_{k=1}^{\infty} \{N(i) \geq m, T_i = k\}.$$

For $k \geq 1$, if $\mathbb{P}(T_i = k \mid X_0 = i) > 0$ then using definition of conditional probability and $N(i)$ we have

$$\begin{aligned}
&\mathbb{P}(N(i) \geq m, T_i = k \mid X_0 = i) && (1.5.5)\\
&= \mathbb{P}(N(i) \geq m \mid T_i = k, X_0 = i)\mathbb{P}(T_i = k \mid X_0 = i)\\
&= \mathbb{P}(N(i) \geq m | X_k = i, X_{k-1} \neq i, \ldots, X_1 \neq i, X_0 = i)\mathbb{P}(T(i) = k \mid X_0 = i)\\
&= \mathbb{P}(N(i) \geq m - 1)\mathbb{P}(T(i) = k \mid X_0 = i). && (1.5.6)
\end{aligned}$$

The last equality is a consequence of the Markov Property (see Exercise **??** and definition of $N(i)$). Intuitively, the memoryless property of the Markov chain implies that the proability of having at least m visits to the state $i \in S$ given that there is already one visit to state i by time k is the same as probability of starting at i and having at least $m - 1$ visits to state $i \in S$ (See Exercise **??**).

For $\mathbb{P}(T_i = k \mid X_0 = i) = 0$ then $\mathbb{P}(N(i) \geq m, T_i = k \mid X_0 = i) = 0$. So, even in this case trivially,

$$\mathbb{P}(N(i) \geq m, T_i = k \mid X_0 = i) = \mathbb{P}(N(i) \geq m - 1 \mid X_0 = i)\mathbb{P}(T_i = k \mid X_0 = i).$$

Using this and (1.5.5) we have,

$$\mathbb{P}(N(i) \geq m \mid X_0 = i) = \sum_{k=1}^{\infty} \mathbb{P}(N(i) \geq m, T_i = k \mid X_0 = i) \tag{1.5.7}$$

$$= \sum_{k=1}^{\infty} \mathbb{P}(N(i) \geq m-1 \mid X_0 = i)\mathbb{P}(T_i = k \mid X_0 = i) \tag{1.5.8}$$

$$= \mathbb{P}(N(i) \geq m-1 \mid X_0 = i) \sum_{k=1}^{\infty} \mathbb{P}(T_i = k \mid X_0 = i) \tag{1.5.9}$$

$$= \mathbb{P}(N(i) \geq m-1 \mid X_0 = i)\mathbb{P}(T_i < \infty \mid X_0 = i) \tag{1.5.10}$$

An inductive argument then shows that, for all $m \geq 1$,

$$\mathbb{P}(N(i) \geq m \mid X_0 = i) = [\mathbb{P}(T_i < \infty \mid X_0 = i)]^m$$

■

We are now ready to state and prove the dichotomy result mentioned at the start of this section.

PROPOSITION 1.5.4. *Let X_n be a Markov chain on S with transition matrix $\mathbf{P}$ and initial distribution μ.*

(a) A state $i \in S$ is recurrent if and only if $\mathbb{P}(T_i < \infty \mid X_0 = i) = 1$.

(b) A state $i \in S$ is transient if and only if $\mathbb{P}(T_i < \infty \mid X_0 = i) < 1$.

In particular a state $i \in S$ is either transient or recurrent.

Proof: The proof follows readily from Lemma 1.5.3. The event

$$\{N(i) = \infty\} = \bigcap_{m=1}^{\infty} \{N(i) \geq m\}.$$

Since the right hand side above is an intersection of decreasing sequence of events.

$$\mathbb{P}(N(i) = \infty \mid X_0 = i) = \lim_{m \to \infty} \mathbb{P}(\{N(i) \geq m\} \mid X_0 = i)$$

Furtehr note that,

$$\lim_{m \to \infty} [\mathbb{P}(T_i < \infty \mid X_0 = i)]^m = \begin{cases} 1 & \text{if } P(T_i < \infty \mid X_0 = i) = 1 \\ 0 & \text{if } P(T_i < \infty \mid X_0 = i) < 1 \end{cases}$$

Now using (1.5.4) from Lemma 1.5.3 and the above we have Since the right hand side above is an intersection of decreasing sequence of events, using (1.5.4) we have

$$\mathbb{P}(N(i) = \infty \mid X_0 = i) = \begin{cases} 1 & \text{if } \mathbb{P}(T_i < \infty \mid X_0 = i) = 1 \\ 0 & \text{if } \mathbb{P}(T_i < \infty \mid X_0 = i) < 1 \end{cases} \tag{1.5.11}$$

By definition of recurrence and transience and (1.5.11) we have the result. ∎

REMARK 1.5.5. (*Criteria for Recurrence and Transience*) As a consequence of the preceeding proposition, one way to decide whether a state $i \in S$ is recurrent or not is to check if $P_i(T_i < \infty) = 1$. That is, if a chain beginning in a state is guaranteed to return to that state (at some later point in time) the state is recurrent. If there is some chance the chain will not return to that state, the state is transient. Checking this condition may sometimes prove difficult so we shall next try to propose another equivalent criteria.

Claim: $i \in S$ is recurrent if and only if

$$\sum_{n=0}^{\infty} p_{ii}^{[n]} = \infty. \tag{1.5.12}$$

Proof of Claim: First we observe that a state $i \in S$ is transient is equivalent to $\mathbb{E}[N(i) \mid X_0 = i] < \infty$. This follows from (1.5.11) and equivalent definition of expectation of $\mathbb{W}$-valued random variables (see Exercise **??**) since

$$\begin{aligned} \mathbb{P}(N(i) < \infty \mid X_0 = i) = 1 &\iff \mathbb{P}(T_i < \infty \mid X_0 = i) < 1 \\ &\iff \sum_{m=1}^{\infty} [\mathbb{P}(T_i < \infty \mid X_0 = i)]^m < \infty \\ &\iff \sum_{m=1}^{\infty} \mathbb{P}(N(i) \geq m \mid X_0 = i) < \infty \qquad (1.5.13) \\ &\iff \mathbb{E}[N(i) \mid X_0 = i] < \infty. \qquad (1.5.14) \end{aligned}$$

Next we will invoke a result on convergence of expectations of non-negative random variables, and obtain

$$\begin{aligned}\mathbb{E}[N(i) \mid X_0 = i] &= \mathbb{E}[\sum_{n=1}^{\infty} 1(X_n = i) \mid X_0 = i] \\ &= \sum_{n=1}^{\infty} \mathbb{E}[1(X_n = i) \mid X_0 = i] \\ &= \sum_{n=1}^{\infty} \mathbb{P}(X_n = i \mid X_0 = i) \\ &= \sum_{n=1}^{\infty} p_{ii}^{[n]}. \end{aligned} \tag{1.5.15}$$

Thus (1.5.13) and (1.5.15) establish (1.5.12). ■.

We conclude with and example on the behaviour of Simple random walk on the integer lattice.

EXAMPLE 1.5.6. Let X_n be a Markov chain on $S = \mathbb{Z}$ with tranistion matrix $\mathbf{P}$ given by

$$p_{ij} = \begin{cases} p & \text{if } j = i+1 \\ 1-p & \text{if } j = i-1 \\ 0 & \text{otherwise,} \end{cases}$$

where $0 < p < 1$. Suppose $P(X_0 = 0) = 1$. Let $i \in \mathbb{Z}$ and $j \in \mathbb{Z}$ with $i < j$ and $j - i = M$. Then

$$p_{ij}^{[M]} = \prod_{k=i}^{j-1} p_{kk+1} = p^M > 0 \text{ and } p_{ji}^{[M]} = \prod_{k=i+1}^{j} p_{kk-1} = (1-p)^M > 0$$

Therefore $i, j \in S$ then $i \leftrightarrow j$ and the chain is irreducible.

We now calculate probabilities at state 0 for all time n. At each time step the chain moves up by one or moves down by one, consequently the chain cannot return to its starting in odd number of steps. For it to return in any even number of steps, say $2n$ steps, it must have moved up by one in n steps and moved down by one in n steps. These can happen in one of $\binom{2n}{n}$ permutations and each such sequence has probability $p^n(1-p)^n$. So as in Example **??**, we have for any $i \in S$,

$$p_{ii}^{[2n]} = \binom{2n}{n} p^n (1-p)^n = \frac{2n!}{n!n!} p^n (1-p)^n.$$

Sterling's formula implies that there is a $c_1 > 0$ such that for all $n \geq 1$

$$\frac{1}{c_1} n^{n+\frac{1}{2}} e^{-n} \leq n! \leq c_1 n^{n+\frac{1}{2}} e^{-n}.$$

Consequently, there is a $c_2 > 0$ such that

$$\frac{1}{c_2} 4^n n^{-\frac{1}{2}} \leq \frac{2n!}{n!n!} \leq c_2 4^n n^{-\frac{1}{2}}.$$

This implies for $n \neq 1$,

$$\frac{1}{c_2} [4p(1-p)]^n n^{-\frac{1}{2}} \leq p_{ii}^{[2n]} \leq c_2 [4p(1-p)]^n n^{-\frac{1}{2}}.$$

When $p = \frac{1}{2}$ then $p_{ii}^{[2n]} \geq \frac{1}{c_2}[4p(1-p)]^n n^{-\frac{1}{2}} = \frac{1}{c_2} n^{-\frac{1}{2}}$ and when $0 < p < 1$, $p \neq \frac{1}{2}$ then $p_{ii}^{[2n]} \leq \frac{1}{c_2} \alpha^n n^{-\frac{1}{2}}$ with $0 < \alpha = 4p(1-p) < 1$. So it is easy to see from common facts about infinite sums that

$$\sum_{k=0}^{\infty} p_{ii}^{[k]} < \infty \iff p \neq \frac{1}{2}.$$

Hence by (1.5.12), $i \in S$ is a recurrent state if and only if $p = \frac{1}{2}$ or in other words, the Markov chain X_n is recurrent if $p = \frac{1}{2}$ and transient otherwise. ■

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