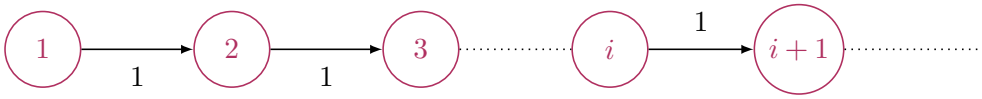
We saw examples of Markov chains that have multiple stationary distributions and those that have a unique stationary distribution. When S is countable there are many examples of a Markov chains that do not have a stationary distribution.

EXAMPLE 1.6.3. Let X_n be a Markov chain on state space $\mathbb{N}$ with transition probabilities $p_{ij} = 1$ if $j = i+1$ (and $p_{ij} = 0$ otherwise). In other words, when $X_n = j$ then $X_{n+1} = j+1$ with probability one. This chain may be represented by the graph



If π were a stationary distribution for X_n then it would have to be that $\pi(\{i+1\}) = \pi(\{i\})$ for all $i \geq 1$. But since $\sum_{k=1}^{\infty} \pi(\{k\}) = 1$ this is not possible. Therefore this Markov chain has no stationary distribution. ■

We now state some basic properties on stationary distributions and limiting distributions.

PROPOSITION 1.6.4. *Let X_n be a Markov Chain with transition matrix P on state space S.*

(a) Let $n \in \mathbb{N}$. Suppose π is a stationary distribution of X_n then $X_0 \stackrel{d}{=} \pi$ implies $X_n \stackrel{d}{=} \pi$.

(b) Suppose S is a fininte state space. Then any limiting distribution must be stationary. That is, suppose for some $i \in S$,

$$\lim_{n\to\infty} p_{ij}^{[n]} = \pi(j) \text{ for all } j \in S,$$

then π is a stationary distribution for X_n.

(c) Suppose S is a finite state space and P is doubly stochastic, (i.e $\sum_{i\in S} p_{ij} = 1$ for all $j \in S$) then the uniform distribution on S is stationary. That is, if $\pi(i) = \frac{1}{|S|}$ for all $i \in S$, then π is a stationary distribution for X_n.

Proof :

(a) As π is stationary, this is immediate from (1.3.1) after an inductive argument.

(b) First let us verify under the hypothesis π is a Probability on S. Clearly, $0 \leq \pi(j) \leq 1$ for all $j \in S$.

$$\sum_{j\in S} \pi(j) = \sum_{j\in S} \lim_{n\to\infty} p_{ij}^{[n]} = \lim_{n\to\infty} \sum_{j\in S} p_{ij}^{[n]} = 1$$

where the limit and sum may be freely interchanged because S is finite. To verify that π is stationary we first observe that for any $n \geq 1$, by (1.1.9) with $r = n-1$ and $s = 1$ we have

$$p_{ij}^{[n]} = \sum_{k \in S} p_{ik}^{[n-1]} p_{kj}.$$

Letting $n \to \infty$ we have, for all $j \in S$,

$$\pi(j) = \sum_{k \in S} \pi(k) p_{kj}.$$

Hence π is a stationary distribution.

(c) Let π be a probability on S. If $|S| = N$ and $\pi(i) = \frac{1}{N}$ for all $i \in S$, then, for each $j \in S$, observe that

$$\sum_{i \in S} \pi(i) p_{ij} = \frac{1}{N} \sum_{i \in S} p_{ij} = \frac{1}{N} = \pi(j).$$

Also $\sum_{i \in S} \pi(i) = 1$. So π is a stationary distribution of X_n. ■

EXAMPLE 1.6.5. (Birth-death chains) Let X_n be a Markov chain on the state space $S = \{l, l+1, \ldots, u-1, u\}$, with $-\infty < l < u < \infty$ and 'tridiagonal' probability transition matrix given by

$$P = \begin{bmatrix} z_l & p_l & 0 & \cdots & \cdots & 0 \\ m_{l+1} & z_{l+1} & p_{l+1} & \ddots & \vdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & m_{u-1} & z_{u-1} & p_{u-1} \\ 0 & \cdots & \cdots & 0 & m_u & z_u \end{bmatrix}.$$

Assume that m_i and p_i are strictly positive where defined. If the chain has a stationary distribution π, then it is not hard to see that

$$\pi(l+i) = c \frac{p_{l+i-1} p_{l+i-2} \cdots p_l}{m_{l+i} m_{l+i-1} \cdots m_{l+1}}.$$

The fact that π is a probability on S will imply that

$$\pi(l) = c = \frac{1}{1 + \sum_{k=l+1}^{u} \frac{p_{k-1} p_{k-2} \cdots p_l}{m_k m_{k-1} \ldots m_{l+1}}}.$$

These chains, where $S \subset \mathbb{Z}$ and $p_{ij} = 0$ if $|i-j| > 1$, are known as Birth-death chains. These are used to model biological systems. ■

We shall conclude this discussion with a statement on convergence to stationary for irreducible and aperiodic Markov Chains.

THEOREM 1.6.6. *Let X_n be a Markov chain on S with transition matrix $\mathbf{P}$ and initial distribution μ. Assume that X_n is aperiodic and irreducible. If X_n has a stationary disrtibution π and if $P(X_0 = i) > 0$, then*

$$\lim_{n\to\infty} p_{ij}^n = \pi(j), \tag{1.6.6}$$

for all $j \in S$.

REMARK 1.6.7.

1. Generalisations: If the chain in Theorem 1.6.6 was periodic with period $d > 1$, it can be shown that there exists an integer r with $0 \leq r < d$ such that

$$\lim_{m\to\infty} p_{ij}^{md+r} = d\pi(j), \tag{1.6.7}$$

for all $j \in S$.

2. Let X_n be a Markov chain on S with transition matrix $\mathbf{P}$ and initial distribution μ. Supppose X_n is a irreducible. It can have atmost one stationary distribution π.

We have already seen it is true when $\mid S \mid < \infty$. Otherwise this can be reasoned as follows. Assume that X_n has two stationary distributions π_1, π_2. Consider a Markov chain $\widetilde{X}_n$ on state space S with transition matrix $\widetilde{\mathbf{P}} = \frac{I+\mathbf{P}}{2}$ where I is the identity matrix. Note that $\widetilde{X}_n$ is an aperiodic and irreducible Markov chain with stationary distributions π_1 and π_2 (verify this). Now Theorem 1.6.6 applied to the chain $\widetilde{X}_n$ implies that $\pi_1 = \pi_2$.

3. (Ergodic Theorem) Let X_n be a Markov chain on S with transition matrix $\mathbf{P}$ and initial distribution μ. Assume that X_n is irreducible and aperiodic. For $i, j \in S$, we have

$$\frac{\sum_{k=1}^n p_{ij}^k}{n} \to \pi(j) \quad a.e. \tag{1.6.8}$$

This is known as the Ergodic Theorem for Markov Chains.

1.7 RANDOM WALK ON WEIGHTED GRAPHS

Let $G = (S, E)$ be a connected graphs. That is let S be a countable vertex set. Let

$$E \subset \{\{i, j\} \mid i, j \in V\}$$

be a set of edges for the graph. Let $\mu : E \to (0, \infty)$. As shorthand notations we define $\mu_{ij} = \mu(\{i, j\})$ for each edge $\{i, j\} \in E$ and $\mu_{ij} = 0$ if $\{i, j\}$ is not an edge. Note the edges are unordered pairs so that $\mu_{ij} = \mu_{ji}$. For each vertex $i \in V$ we further define

$$\mu_i = \sum_{\{i,j\} \in E} \mu_{ij}$$

Suppose that

$$0 < \mu_i < \infty.$$

for each $i \in V$, then we may define a matrix $\mathbf{P}$ with entries $p_{ij} = \frac{\mu_{ij}}{\mu_i}$ for all $i, j \in S$. Since each $p_{ij} \geq 0$ and since

$$\begin{aligned} \sum_{j \in S} p_{ij} &= \sum_{j \in S} \frac{\mu_{ij}}{\mu_i} \\ &= \frac{1}{\mu_i} \sum_{j \in S} \mu_{ij} \\ &= 1 \end{aligned}$$

then $\mathbf{P}$ is the transition matrix for a Markov chain on S. Therefore on any weighted graph G with a described initial distribtion X_0 on its vertices there is a canonical Markov chain X_n with state space S and transition matrix given by $\mathbf{P}$. Such a Markov chain is referred to as *Random walk on the weighted graph.*

PROPOSITION 1.7.1. *Suppose X_n is a random walk on a connected weighted graph G with weight function μ. If $(S) < \infty$ then the walk is recurrent.*

Proof. Sketch and the proof is not in syllabus

Let $j_0 \in S$ define $h : S \to [0, 1]$ be given by $h(i) = \mathbb{P}(T_{j_0} < \infty \mid X_0 = i)$. Let $\phi = 1 - \phi$. It is easy to see that as $\mu(S) < \infty$ and using the properties of ϕ

$$\sum_{i \in S} \sum_{j \in S} \mu_{ij} (\phi(i) - \phi(j))^2 = 0.$$

This implies that ϕ is constant but $\phi(j_0) = 0$. Therefore $h(i) = 1$ for all $i \in S$. So the walk is recurrent.

■

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