**Wednesday, September 25th, 2025**
*Time: 2:10-3:00pm*

1. Let $\{X_n\}_{n\geq 0}$ be a Markov chain on $S$ with transition matrix $\mathbf{P}$ and initial distribution $\eta$. A probability measure $\pi$ on $S$ is said to be *reversible* w.r.t $\mathbf{P}$ if

$$\pi(\{i\})p_{ij} = \pi(\{j\})p_{ji} \quad \text{for every } i, j \in S.$$

Show that if $\pi$ is reversible then it is a stationary distribution for $\{X_n\}_{n\geq 0}$.

2. (*Random walk on the weighted graph* $(G, \mu)$.) Let $G = (S, E)$ be a connected graph. Assume $0 <\mid S \mid< \infty$ and $E \subset \{\{i, j\} \mid i, j \in S\}$ be a set of edges for the graph. Let $\mu : E \to (0, \infty)$ be the weight function. As shorthand notations we define $\mu_{ij} = \mu(\{i, j\})$ for each edge $\{i, j\} \in E$ and $\mu_{ij} = 0$ if $\{i, j\}$ is not an edge. Note that the edges are unordered pairs so that $\mu_{ij} = \mu_{ji}$. For each vertex $i \in S$ we further define

$$\mu_i = \sum_{\{i,j\}\in E} \mu_{ij}$$

Define the transition matrix $\mathbf{P}$, for the Markov chain $\{X_n\}_{n\geq 0}$ on $S$ with entries

$$p_{ij} = \frac{\mu_{ij}}{\mu_i}$$

for all $i, j \in S$. Find the *reversible* probability measure $\pi$ for $\{X_n\}_{n\geq 0}$.

3. Let $\{X_n\}_{n\geq 0}$ be a Markov chain on $S$ with transition matrix $\mathbf{P}$ and initial distribution $\eta$. Let $\pi$ be a reversible probability measure for $\{X_n\}_{n\geq 0}$. Show that for any $n \geq 1$ and $i, i_1, i_2, \ldots i_{n-1}, j \in S$ we have

$$\begin{aligned}&\pi(\{i\})\mathbb{P}(X_1 = i_1, X_2 = i_2, \ldots, X_{n-2} = i_{n-2}, X_{n-1} = i_{n-1}, X_n = j \mid X_0 = i)\\ &= \pi(\{j\})\mathbb{P}(X_1 = i_{n-1}, X_2 = i_{n-2}, \ldots, X_{n-2} = i_2, X_{n-1} = i_1, X_n = i \mid X_0 = j)\end{aligned}$$

Draw a schematic picture for $n = 3$ case and explain the term reversible in words.

**(5 points) for on time submission**