Due: Wednesday, July 29th, 2026, 8am
Problems to be turned in: 4,5

Set up: Let $N \in \mathbb{N}$. Consider

$$\Omega_N = \{\omega = (\omega_1, \omega_2, \ldots, \omega_N) : \omega_i \in \{-1, +1\}\}$$

equipped with the uniform distribution, denoted by $\mathbb{P} \equiv \mathbb{P}_N$ given by

$$\mathbb{P}(A) = \frac{|\, A \,|}{2^N},$$

for any $A \subset \Omega_N$. For $1 \leq k \leq N$, let $X_k : \Omega_N \to \{-1, 1\}$ be given by

$$X_k(\omega) = \omega_k$$

and for $1 \leq n \leq N$, let $S_n : \Omega_N \to \{-1, 1\}$ be given by

$$S_n(\omega) = \sum_{k=1}^{n} X_k(\omega)$$

and $S_0 = 0$.

1. Show that $\{X_i\}_{1 \leq i \leq N}$ are indepdendent and identically distributed with distribution given by $\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = \frac{1}{2}$.
2. Prove that an increment $S_m - S_k$ for $0 < k < m \leq N$ has the same distribution as S_{m-k}.
3. Prove that increment $S_m - S_k$ is independent of $S_n - S_l$ for $0 < l < n < k < m \leq N$
4. Show that the mode of the distribution is achieved in the middle, i.e.

   $$\mathbb{P}(S_{2n} = 0) = \mathbb{P}(S_{2n-1} = 1) = \binom{2n}{n} \frac{1}{2^{2n}}.$$

   Show that there exists $N_0 \in \mathbb{N}$ and $C_1, C_2 > 0$ such that for all $n \geq N_0$.

   $$\frac{C_1}{\sqrt{n}} \leq \mathbb{P}(S_{2n} = 0) \leq \frac{C_2}{\sqrt{n}}$$

   and

   $$\frac{C_1}{n\sqrt{n}} \leq \mathbb{P}(S_{2n} = 0) - \mathbb{P}(S_{2n+2} = 0) \leq \frac{C_2}{n\sqrt{n}}$$

5. Assume for that $\{a_i\}_{i=1}^{n-1}$ are a set of integers such that $\mathbb{P}(S_{n-1} = a_{n-1}, \ldots, S_1 = a_1) > 0$. Show that

   $$P(S_n = a_n \mid S_{n-1} = a_{n-1}, \ldots, S_1 = a_1) = \mathbb{P}(S_n = a_n \mid S_{n-1} = a_{n-1})$$

6. Let for $0 < k \leq N$, $a \in \mathbb{Z}, \mathbb{P}(S_k = a) > 0$. Prove that for $0 < k < m \leq N$,

   $$\mathbb{P}(S_m = b \mid S_k = a) = \mathbb{P}(S_{m-k} = b - a)$$

   for $b \in \mathbb{Z}$.