Due October 14th, 2026 *Time and Problems due: will be mentioned in class*

1. Consider a martingale $\{Z_n\}_{n\geq 1}$, $N \geq 1$ and $a < b$. Let $U_N(a,b)$ be the number of upcrossings of $[a,b]$ by Z_n. Show that
$$\mathbb{E}[U_N(a,b)] \leq \frac{1}{b-a}\left(\mathbb{E} \mid Z_n \mid + \mid a \mid\right)$$

2. Consider a martingale where Z_n can take on only the values 2^{-n-1} and $1-2^{-n-1}$, each with probability $\frac{1}{2}$.
   (a) Given that Z_n, conditional on Z_{n-1}, is independent of $Z_{n-2}, Z_{n-3}, \ldots, Z_1$ find $\mathbb{P}(Z_n = \cdot \mid Z_{n-1})$ for each n so that the martingale condition is satisfied.
   (b) Show that $\mathbb{P}(\sup_{n\geq 1} Z_n \geq 1) = \frac{1}{2} \neq 0 = \mathbb{P}(\bigcup_{n\geq 1}\{Z_n \geq 1\})$
   (c) Show that for all $a > 0$ and $0 < \epsilon < a$, $\mathbb{P}(\sup_{n\geq 1} Z_n \geq a) \leq \frac{E[Z_1]}{a-\epsilon}$.

3. Let $\{X_n\}_{n\geq 1}$ be independent with
$$X_n = \begin{cases} 1 & \text{with probability } \frac{1}{2n} \\ 0 & \text{with probability } 1-\frac{1}{n} \\ -1 & \text{with probability } \frac{1}{2n} \end{cases}$$
For $n \geq 1$,
$$Y_n = \begin{cases} X_n & \text{if } n = 1 \text{ or } Y_{n-1} = 0, n \geq 2 \\ nY_{n-1} \mid X_n \mid & \text{if } Y_{n-1} \neq 0, n \geq 2. \end{cases}$$
   (a) Show that Y_n is a martingale.
   (b) Find $\lim_{n\to\infty} \mathbb{P}(Y_n = 0)$.
   (c) Decide if Martingale convergence theorem applies and whether Y_n converges to some Y as $n \to \infty$ with probability 1.

4. (Branching Process) For $n \geq 1$, let $\{\xi_k^n\}_{k\geq 1}$ be i.i.d. ξ with range of X being in $\mathbb{N}\cup\{0\}$, $m = \mathbb{E}[\xi]$ and $\sigma^2 = \mathrm{Var}(\xi)$. Define a sequence Z_n by, $Z_1 = 1$ and for $n \geq 2$,
$$Z_n = \sum_{i=1}^{Z_{n-1}} \xi_i^n,$$
with the sum being interpreted as 0 if $Z_{n-1} = 0$.
   (a) Show that $\frac{Z_n}{m^n}$ is a martingale.
   (b) Let $m > 1$. Find $\mathbb{E}[Z_n]$ and using the law of total variance appropriately find $\mathrm{Var}(Z_n)$. Apply the second moment method show that $\mathbb{P}(Z_n \geq 1 \text{for all } n \geq 1) > 0$.
   (c) Let $m = 1$. Show that there is a random variable Z such that Z_n converges to Z with probability one, as $n \to \infty$. What can you say about the distribution of Z ?

5. Let S_n be the simple symmetric walk on $\mathbb{Z}^d$ starting at 0. Let $h : \mathbb{Z}^d \to \mathbb{R}$ be harmonic, i.e.
$$h(x) = \frac{1}{2d}\left(\sum_{i=1}^{d} h(x+e_d) + h(x-e_d)\right),$$
whenever $x \in \mathbb{Z}^d$, where $\{e_i : 1 \leq i \leq d\}$ are the standard basis for $\mathbb{Z}^d$.
   (a) For $d = 1, 2, 3$ provide three examples of harmonic functions h.
   (b) Show that $\{Z_n\}_{n\geq 1}$ given by $Z_n = h(S_n)$ for $n \geq 1$ is a martingale.