Due October 21,2026

Time and Problems due: will be mentioned in class

1. (*Monkey typing "ABRACADABRA"*) A monkey draws independently and uniformly from the English alphabet $S = \{A, B, C, D, \dots Z\}$. Let U_i be the result of the i-th draw. Let $\mathcal{A}_n$ be the observable events from $U_1, U_2, \dots, U_n$. Let the random time τ be defined as

$$\tau = \min\{n \geq 11 : (U_{n-10}, U_{n-9}, \dots U_n) = (A, B, R, A, C, A, D, B, R, A)\}$$

   (a) Show that τ is a stopping time and $\mathbb{E}[\tau] < \infty$

   (b) Find a martingale $(M_n)_{n\geq 1}$ such that

      i. $M_1 = 0$ and $M_\tau = \tau - 26^{11} - 26^4 - 26$

      ii. there is a $C > 0$ such that $\mid M_{n+1} - M_n \mid \leq C$ for all $n \geq 1$

   (c) Show that $\mathbb{E}[\tau] = 26^{11} + 26^4 + 26$

2. Let $\{M_k\}_{1\leq k\leq n}$ be a martingale and let $\mathcal{A}_n$ be events observable till time n.

   (a) Let $n \geq 1$ and $2 \leq k \leq n$, $D_k = M_k - M_{k-1}$. Assume $|D_k| \leq c$. Let $\lambda \in \mathbb{R}$.

      i. Show that for any $d \in [-c, c]$

$$e^{\lambda d} \leq \frac{c-d}{2c} e^{-\lambda c} + \frac{c+d}{2c} e^{\lambda c}, \qquad |d| \leq c.$$

      ii. Conclude that

$$\mathbb{E}\big[e^{\lambda D_k} \mid \mathcal{A}_n\big] \leq e^{\lambda^2 c^2/2}.$$

   (b) Let τ be a stopping time. Show that

$$E[\tau] < \infty \qquad \text{and there exists } c > 0 \text{ such that } \mid M_n - M_{n-1} \mid \leq c$$

   implies

$$\lim_{n\to\infty} \mathbb{E}[M_n \mid \tau > n]\mathbb{P}(\tau > n) = 0 \qquad \text{and } \mathbb{E}[M_\tau] < \infty$$