Due October 28, 2026

Time and Problems due: will be mentioned in class

1. Let $\lambda < \frac{n}{2}$. Consider Erdös-Rényi Graph $G(n, \frac{\lambda}{n})$ and let Y_n be the number of isolated vertices.
   (a) Show that $\mathbb{E}[Y_n] = n(1 - \frac{\lambda}{n})^{n-1}$
   (b) Show that
   $$(ne^{-\lambda})(1 - \frac{\lambda^2}{n}) \leq \mathbb{E}[Y_n] \leq (ne^{-\lambda})(1 + \frac{2\lambda}{n}).$$
   (c) Conclude that if $\lambda - \log(n) \to -\infty$ as $n \to \infty$ then $\mathbb{E}[Y_n] \to \infty$ as $n \to \infty$.

2. Assume Cayley's Theorem: The number of labeled trees of size n is equal to n^{n-2}. Consider Erdös-Rényi Graph $G(n, \frac{\lambda}{n})$ for $\lambda > 0$. For $k \geq 1$, let X_k be the number of occupied trees of size equal to k on the vertex set $[n]$ that cannot be extended to a tree of larger size[1].
   (a) Show that
   $$\mathbb{E}[X_k] = \binom{n}{k} k^{k-2} \left(\frac{\lambda}{n}\right)^{k-1} \left(1 - \frac{\lambda}{n}\right)^{k(n-k)}$$
   (b) Using Stirling's approximation, show that for $\lambda = a\log(n)$ with $a > \frac{1}{2}$ and for all $k \geq 5$
   $$\mathbb{E}[X_k] \leq n^{1-\frac{k}{4}}$$

3. Consider Erdös-Rényi Graph $G(n, \frac{\lambda}{n})$ and let I_i be the indicator that vertex i is solated. Let $Y_n = \sum_{i=1}^n I_i$ be the number of isolated vertices. Assume $\lambda - \log(n) \to t$ for some $t \in \mathbb{R}$ as $n \to \infty$.
   (a) Show that $\mathbb{E}[Y_n] \to e^{-t}$ as $n \to \infty$.
   (b) Suppose for $r \geq 1$, $(Y_n)_r = \prod_{k=0}^{r-1} (Y_n - k)$.
      i. (Not due) For $r \geq 1$, show that $\mathbb{E}[(Y_n)_r] = \sum_{i_1, i_2, \ldots i_2 = 1}^{n} \mathbb{P}(I_{i_1} = 1, \ldots I_{i_n} = 1)$, where the sum is over distinct indices.
      ii. Use (i) and show that $\mathbb{E}[(Y_n)_r] = \frac{n!}{(n-r)!}\mathbb{P}(I_1 = 1, I_2 = 1, \ldots I_r = 1)$
      iii. Show $\mathbb{P}(I_1 = 1, I_2 = 1, \ldots, I_r = 1) = n^{-r}(E[Y_n])^r(1 + a_n)$ where $a_n \to 0$ as $n \to \infty$.
      iv. Show that $\mathbb{E}(Y_n)_r \to e^{-tr}$ as $n \to \infty$.
      v. (Not due) Conclude that Y_n converges in distribution to Poisson (e^{-t}).

The above treatment is based on Section 5.3 in : R. van der Hofstad. Random Graphs and Complex Networks. Volume One. Volume 43 in Cambridge Series in Statistical and Probabilistic Mathematics (2017) ISBN 978-1-107-17287-6

[1] Thus, each tree that is counted in X_k has size precisely equal to k and there are no edges between the k vertices in the tree and $n - k$ vertices outside it. Further there are precisely $k - 1$ edges between the k vertices that are such that these vertices together with the edges form a tree.