Due: Wednesday, August 5th, 2026, 6pm
Problems to be turned in: to be announced at August 6th, in class.

Set up: Let $N \in \mathbb{N}$. Consider the probability space $(\Omega_N, \mathcal{F}, \mathbb{P})$ given by

$$\Omega_N = \{-1,1\}^N, \qquad \mathcal{F} = \mathcal{P}(\Omega), \qquad \mathbb{P}(A) = \frac{|\,A\,|}{2^N}, \text{ for any } A \subset \Omega_N.$$

For $1 \le k \le N$, let $X_k : \Omega_N \to \{-1,1\}$ be given by

$$X_k(\omega) = \omega_k$$

and for $1 \le n \le N$, let $S_n : \Omega_N \to \{-1,1\}$ be given by

$$S_n(\omega) = \sum_{k=1}^{n} X_k(\omega), \text{ and } S_0 = 0.$$

For $n \in \{0,1,\ldots,N\}$, let $\mathcal{A}_n$ be the events that are observable by time n.

1. Show that $\mathcal{A}_n$ is closed under complimentation and intersections. Further establish that
$$\{\emptyset, \Omega\} = \mathcal{A}_0 \subset \mathcal{A}_1 \subset \mathcal{A}_2 \subset \ldots \mathcal{A}_n \subset \mathcal{A}_{n+1} \ldots \subset \mathcal{A}_N = \mathcal{F}$$
2. For $n \in \{0,1,\ldots,N-1\}$. Suppose $A_n \in \mathcal{A}_n$ Then show that
$$\mathbb{P}(A_n \cap \{X_{n+1} = 1\}) = \frac{1}{2}\mathbb{P}(A_n), \qquad \mathbb{E}[X_{n+1} 1_{A_n}] = 0.$$
3. Let $T : \Omega \to \{1,2,\ldots,N\}$ be a stopping time. Show, via a direct proof, that $\mathbb{E}(S_T) = 0$.
4. Let $-\infty < a < 0 < b < \infty, a, b \in \mathbb{Z},$,
$$\sigma_a = \min\{k \ge 1 : S_k = a\} \qquad \text{and} \qquad \sigma_b = \min\{k \ge 1 : S_k = b\}.$$
   (a) Let $\tau_N = \min\{\sigma_a, \sigma_b, N\}$. Show that τ_N is a Stopping time.
   (b) Show that
   $$\mathbb{E}(S_{\tau_N}) = a\mathbb{P}(\sigma_a < \sigma_b, \sigma_a \le N) + b\mathbb{P}(\sigma_b < \sigma_a, \sigma_b \le N) + \mathbb{E}(S_N 1(\min\{\sigma_a \sigma_b\} > N))$$
   and
   $$\mathbb{E}(\tau_N) = a^2\mathbb{P}(\sigma_a < \sigma_b, \sigma_a \le N) + b^2\mathbb{P}(\sigma_b < \sigma_a, \sigma_b \le N) + \mathbb{E}(S_N^2 1(\min\{\sigma_a, \sigma_b\} > N)).$$
   (c) $\mathbb{E}(\tau_N) \to -ab$ as $N \to \infty$.
5. (*Ballot Theorem*) Suppose L and W are candidates contesting the elections to be president of students union. In the ballot, suppose each student voter is equally likely to vote for L and W. THe results are announced and L receives l votes and W receives w votes with $w > l$.
   (a) Let $N = W + L$ and for $1 \le k \le N$ let S_k represents the number of votes received by W minus the number of votes received by L at the counting of k-th ballot.
      i. Show that S_k is a simple symmetric random walk of length N.

ii. Describe the $\mathbb{P}$(that W was leading through out the counting and wins by $w-l$ votes)) in terms of Probabilities involving the random S_k

(b) Find the $\mathbb{P}(S_k > 0, 1 \leq k \leq N \mid S_N = w-l)$ using the following steps:

i. Show that the number of valid random walk paths from $(1,1)$ to $(N, w-l)$ that touch the x-axis is equal to the number of paths valid random walk paths from $(1,-1)$ to $(N, w-l)$.

ii. Show that the number of valid random walk paths from $(1,1)$ to $(N, w-l)$ is $\binom{w+1-1}{l}$.

iii. Show that the number of valid random walk paths from $(1,-1)$ to $(N, w-l)$ is $\binom{w+l-1}{l-1}$.

iv. Conclude that the number of random walk paths from $(1,1)$ to $(N, w-l)$ that do not touch the x-axis is given by $\frac{w-l}{w+l}\binom{w+l}{l}$.

(c) Show that $\mathbb{P}$(that W was leading through out the counting and wins by $w-l$ votes)) $= \frac{w-l}{w+l}$