Due: Wednesday, August 19th, 2026, 610pm

Time and problems to be turned in: to be announced on August 19th in class

1. Let $S = \{0, 1, 2, 3, 4\}$ and the transition matrix be given by

$$\mathbf{P} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

   Let X_n be a Markov chain on S with transition matrix $\mathbf{P}$ and initial distribution $\mathbb{P}(X_0 = 0) = 1$.

   (a) Draw a weighted graph so that the simple random walk on it would be X_n.

   (b) Find a unit vector π such that $\pi P = \pi$. Show that π is the stationary distribution X_n, that is if $X_0 \stackrel{d}{=} \pi$ then $X_n \stackrel{d}{=} \pi$ for all $n \geq 1$.

   (c) Find the $\lim_{n\to\infty} \mathbb{P}(X_n = i)$ for each $i \in S$.

2. Let $0 < p < 1$, $L \geq 1$ and $S = \{0, 1, 2, \ldots, L\}$. Assume $\mathbb{P}(S_0 = 1) = 1$ and X_n be a Markov chain with transition matrix $\mathbf{P} = [p_{ij}]_{i,j\in\mathbb{Z}}$ given by

$$p_{ij} = \begin{cases} p & \text{if } j = i+1, i \neq L \\ 1-p & \text{if } j = i-1, i \neq 0 \\ p & \text{if } j = 0, i = L \\ 1-p & \text{if } j = L, i = 0 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Sketch the graph induced by this Markov chain on the vertex set S.

   (b) Find the stationary distribution π of the chain.

3. Let X_n be a Markov chain on S with transition matrix P and initial distribution μ.

   (a) Let $i \in S$ is a recurrent state. Suppose there is a $j \in S$ and $r, s \in \mathbb{N}$ such that $p_{ij}^r > 0$ and $p_{ji}^s > 0$ then show that $j \in S$ is also a recurrent state.

   (b) Assume that $\mu(\{i\}) > 0$ for all $i \in S$. For $A \subset S$ let $T^A = \min\{n \geq 0 : X_n \in A\}$. For $n \geq 2$, show that

$$\mathbb{P}(T^A = n \mid X_0 = i) = \sum_{j \notin A} p_{ij} \mathbb{P}(T^A = n-1 \mid X_0 = j).$$

   (c) For all $i \in S$ assume that $\mu(\{i\}) > 0$ and suppose for all $i, j \in S$ there is a $r, s \in \mathbb{N}$ such that $p_{ij}^r > 0$ and $p_{ji}^s > 0$. Let $T_i = \min\{n \geq 1 : X_n = i\}$. Show that

$$i \in S \text{ is recurrent} \iff \mathbb{P}(T_i < \infty \mid X_0 = j) = 1 \quad \text{for all } j \in S$$

4. Let $S = \mathbb{Z}^d$ and transition matrix $\mathbf{P} = [p_{ij}]_{i,j \in S}$ be given by

$$p_{ij} = \begin{cases} \frac{1}{2d} & \text{if } \mid i - j \mid = 1 \\ \\ 0 & \text{otherwise} \end{cases}$$

Let X_n be a Markov chain on S with transition matrix $\mathbf{P}$ and initial distribution $\mathbb{P}(X_0 = 0) = 1$. Show that for $d = 2$, $(X_n)_{n \geq 1}$ is recurrent while it is transient for $d = 3$.