Due: Wednesday, September 23rd, 2026
Time and Problems due: will be mentioned in class

1. Let X, Y, Z be discrete random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Suppose that $\mathbb{P}(Y = y) > 0$ and $\mathbb{P}(Z = z) > 0$.
   (a) Show that $\mathbb{E}[XY \mid Y = y] = y\mathbb{E}[X \mid Y = y]$
   (b) If X is independent of Y then $\mathbb{E}[X \mid Y = y] = \mathbb{E}[X]$ and hence $\mathbb{E}[X \mid Y] = \mathbb{E}[X]$

2. Let X, Y, Z be discrete random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Suppose that $\mathbb{P}(Y = y) > 0$ and $\mathbb{P}(Z = z) > 0$.
   (a) Show that $\mathbb{E}[X + Y \mid Y = y] = y + \mathbb{E}[X \mid Y = y]$
   (b) Show that $\mathbb{E}[Y \mid Y = y, Z = z] = y$

3. Let $X \sim$ Uniform $\{1, 2, \ldots, n\}$ be independent of $Y \sim$ Uniform $\{1, 2, \ldots, n\}$. Let $Z = \max(X, Y)$ and $W = \min(X, Y)$.
   (a) Find the joint distribution of (Z, W).
   (b) Find $E[Z \mid W]$.

4. On a suitable probability space $(\Omega, \mathcal{F}, \mathbb{P})$ let S_n be the simple symmetrics random walk on $\mathbb{Z}$. Let $\mathcal{A}_n$ be the events that are observable by time n[2].
   (a) Show that $\xi_n = S_n^2 - n$ is a martingale w.r.t the filtration $\mathcal{A}_n$.
   (b) Find $\{a_n\}_{n\geq 1}$ such that $\eta_n = S_n^3 + a_n S_n$ is a martingale w.r.t filtration $\mathcal{A}_n$.
   (c) Find $\{b_n\}_{n\geq 1}$ such that $\delta_n = S_n^4 + b_n S_n^2 + c_n$ is a martingale w.r.t filtration $\mathcal{A}_n$.
   (d) Let $\{V_k\}_{k\geq 1}$ be a *predictable process*, that is for $c \in \mathbb{R}$,
   $$\{V_k = c\} \in \mathcal{A}_{k-1}.$$
   Then show that
   $$Z_0 = 0 \qquad Z_n = \sum_{k=1}^{n} V_k(S_k - S_{k-1})$$
   is a martingale w.r.t the filtration $\mathcal{A}_n$.

5. An urn contains R red and G green balls. At each time we draw a ball from the urn, then replace it, and add C balls of the colour drawn. Let $\{X_n\}_{n\geq 1}$ be the fraction of green balls after the n-th draw. Show that X_n is a martingale.

[2] We shall say a sequence of random variables, $\{H_n\}$ is a martingale w.r.t. the filtration $\mathcal{A}_n$ if
$$\mathbb{E}[\mid H_n \mid] < \infty \text{ and } \mathbb{E}[H_n \mid S_{n-1}, \ldots, S_1] = H_{n-1}$$