Due: Wednesday, October 7th, 2026
Time and Problems due: will be mentioned in class

1. Let $\Omega = \{-1,1\}^{\mathbb{N}}$. Consider for $k \geq 1$, $X_k : \Omega \to \{-1,1\}$ which are i.i.d. X with distribution given by$\mathbb{P}(X=1) = p = 1 - \mathbb{P}(X=-1)$, for some $0 < p < 1$. For $n \geq 1$, let $S_n = \sum_{k=1}^{n} X_k$ and $S_0 = 0$, Suppose for $a < 0 < b$,
$$T = \min\{n \geq 1 : S_n \notin (a,b)\}.$$
Show that T is a stopping time and $\mathbb{P}(T \geq n) \to 0$ as $n \to \infty$.

2. Consider the graph $\Gamma = (V,E)$ with $V = \mathbb{Z}^2$ and $E = \{\{\mathbf{i},\mathbf{j}\} : \mathbf{i},\mathbf{j} \in V, \mathbf{i} \neq \mathbf{j} \text{ and } \| \mathbf{i}-\mathbf{j} \| \leq \sqrt{2}\}$ with $\| \cdot \|$ denoting the euclidean norm. Consider the standard weight function $\mu \equiv 1$ on E. Let $\{X_n\}_{n\geq 1}$ be the random walk on this weighted graph (Γ, μ) with $X_0 = 0$.
   (a) Show that $M_n = \| X_n \|^2 - \frac{3}{2}n$ for all $n \geq 1$, is a martingale.
   (b) Let $\tau_R = \min\{n \geq 0 : \| X_n \|^2 \geq R^2\}$. Show that
$$\frac{2}{3}R^2 \leq E[\tau_R] \leq \frac{2}{3}(R+\sqrt{2})^2$$

3. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space on which $\{M_n\}_{n\geq 1}$ is a martingale with Range$(M_n) \subset \mathbb{Z}$. Let $T : \Omega \to \mathbb{N}$ be a stopping such that $\mathbb{E}[T] < \infty$. Let $\epsilon > 0$ be given.
   (a) Show that there is a $M > 0$ such that $\sum_{k=M}^{\infty} \mathbb{P}(| M_1 | > k) < \epsilon$.
   (b) Show that there is a $N > 0$ such that $\mathbb{P}(T > N) < \epsilon$.

   Conclude that $\lim_{n\to\infty} \mathbb{E}[| M_1 \mid 1(T > n)] = 0$

4. Let $\{Z_n\}_{n\geq 1}$ be a martingale on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and for some integer m, let $Y_n = Z_{n+m} - Z_m$. Show that $\{Y_n\}_{n\geq 1}$ is a martingale.

5. (*Ex 7.21 DSP Gallagher*)Let $\{X_i\}_{i\geq 1}$ be a sequence of i.i.d. random variables such that $\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = \frac{1}{2}$. Let $Z_1 = -2$ and
$$Z_{n+1} = Z_n(1 + X_n\left(\frac{3n+1}{n+1}\right))$$
   (a) Show that $\{Z_n\}_{n\geq 1}$ is a martingale.
   (b) Let $J = \min\{n \geq 2 : \text{sign}(Z_n) = \text{sign}(Z_{n-1})\}$ be a stopping time. Using induction on n show that
$$\mathbb{P}(Z_n = \frac{(-2)^n}{n} \mid J > n) = 1 \qquad \text{and } \mathbb{P}(Z_n = -\frac{(-2)^n(n-2)}{n^2-n} \mid J = n) = 1$$
   (c) Show that $\mathbb{E} \mid Z_J \mid = \infty$ and $\lim_{n\to\infty} \mathbb{E}[Z_n \mid J > n]\mathbb{P}(J > n) = 0$