1. Random Walk on $\mathbb{Z}$ finite length N

Recall:

Probability Space $(S, \mathcal{F}, \mathbb{P})$

S - sample space, $|S| < \infty$

S - countable .. $\mathbb{N}$, $\mathbb{Z}$

$\mathcal{F}$ - Events $\quad \mathcal{F} = \mathcal{P}(S)$

$\mathbb{P}: \mathcal{F} \longrightarrow [0,1]$

(A1) $\mathbb{P}(S) = 1$

(A2) $\{E_k\}_{k \geq 1} \quad E_i \cap E_j = \phi \quad i \neq j$

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(E_i)$$

Independence

A finite collection of events $A_1, A_2, \ldots, A_n$ is mutually independent if

(4) — $\mathbb{P}(E_1 \cap E_2 \cap E_3 \ldots \cap E_n) = \prod_{i=1}^{n} \mathbb{P}(E_i)$

where $E_i = A_i$ or $A_i^c \quad i = 1, 2, \ldots, n$.

An arbitrary collection of events A_t where $t \in I$ for some index set I is mutually independent if every finite subcollection is mutually independent.

- A, B are independent iff

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B)$$

Conditional Probability

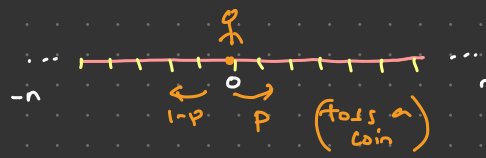
- A, B are two events $\quad \mathbb{P}(B) > 0$

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

Definition of Random Walk

Intuitive : R.W.

- Toss a coin at each step
- if head occurs move right
- if tail occurs move left



S_n = position at time n

Q: - Typical position at time n?

- $P(S_n = 0) = ?$
- Find $x \in [-n, n]$ s.t. $P(S_n = x)$ is maximum?

Notation: $\mathbb{N}$ – natural numbers

$\mathbb{N}_0$ – $\mathbb{N} \cup \{0\}$.

Simple random walk on $\mathbb{Z}$

Fix $N \in \mathbb{N}$.

$$\Omega_N = \{ \omega = (\omega_1, \dots, \omega_N) \mid \omega_i \in \{-1, +1\} \} \equiv \{-1, 1\}^N$$

Define for $1 \le k \le N$

$X_k : \Omega_N \to \{-1, 1\}$ by $X_k(\omega) = \omega_k$

(step of random walk at time k)

$$S_0(\omega) = 0 \; ; \quad S_n(\omega) = \sum_{k=1}^{n} X_k(\omega) \quad 1 \le n \le N$$

(position of walk at time n)

$$\mathcal{F}_N = \{ A \mid A \subseteq \Omega_N \}$$

$$\mathbb{P}_N : \mathcal{F}_N \to [0,1]$$ (Uniform Probability) ($p = \frac{1}{2}$ intuitive)

(1) $$\mathbb{P}_N(A) = \frac{|A|}{2^N} \quad \forall A \subseteq \Omega_N$$

(i.e. any sequence in $\{-1,1\}^N$ has the same probability)

Definition 1.1: Sequence of random variable $\{S_n\}_{n=0}^N$ on Probability space $(\Omega_N, \mathcal{F}_N, \mathbb{P}_N)$ is called a simple symmetric random walk of length N starting at 0.

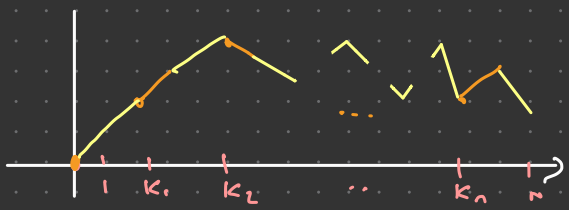
Observations:

Given :- $1 \le k_1 < k_2 \dots < k_n \le N$

$x_1 \quad x_2 \quad \dots \quad x_n$ ($x_i \in \{-1,1\}$)

steps in position

(*) $$\mathbb{P}(X_{k_1} = x_1, \dots, X_{k_n} = x_n) = ? = \frac{2^{N-n}}{2^N} = \frac{1}{2^n}$$



e.g. $x_1 = 1$, $x_2 = -1$, $\vdots$, $x_n = 1$

$$|\{X_{k_1} = x_1, \dots, X_{k_n} = x_n\}| = |\{\omega \in \Omega_N : \omega_{k_1} = x_1, \dots, \omega_{k_n} = x_n\}| = 2^{N-n}$$

- (Ex.) $\mathbb{P}(X_k = 1) = \frac{1}{2} = \mathbb{P}(X_k = -1)$

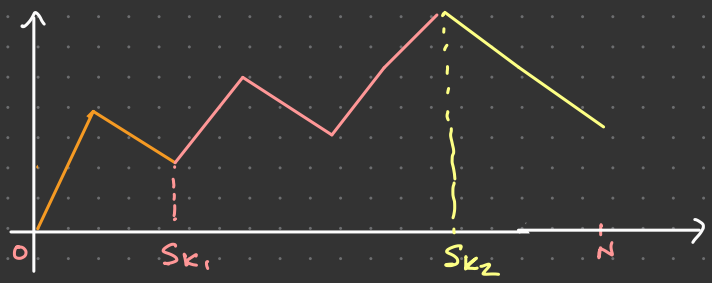
$\forall\ 1 \le k \le N$

[$\because$ $n=1$ in $*$]

$k \neq l$ $\quad \mathbb{P}(X_k = \pm 1, X_l = \pm 1) = \mathbb{P}(X_k = \pm 1)\,\mathbb{P}(X_l = \pm 1)$

(independence) $\quad k \neq l$

- (Ex)



Simple random walk has <u>independent increments</u>

$S_{k_1} - S_0$, $\quad S_{k_2} - S_{k_1}$, $\quad S_N - S_{k_2}$

are independent of each other.

<u>Note:</u>

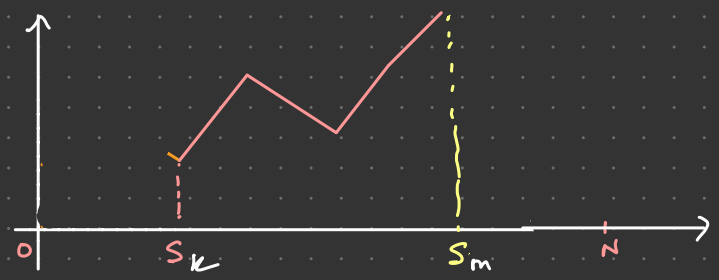
- $S_{k_1} = \sum_{n=0}^{k_1} X_n$
- $S_{k_2} = S_{k_1} + \sum_{n=k_1+1}^{k_2} X_n$

are <u>dependent</u>

- $0 < k < m \le N$

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$$\mathbb{P}(S_m - S_k = a) = \mathbb{P}(S_{m-k} = a)$$



- $S_k = \sum_{n=0}^{k} X_n$
- $S_m = S_k + \sum_{n=k+1}^{m} X_n$

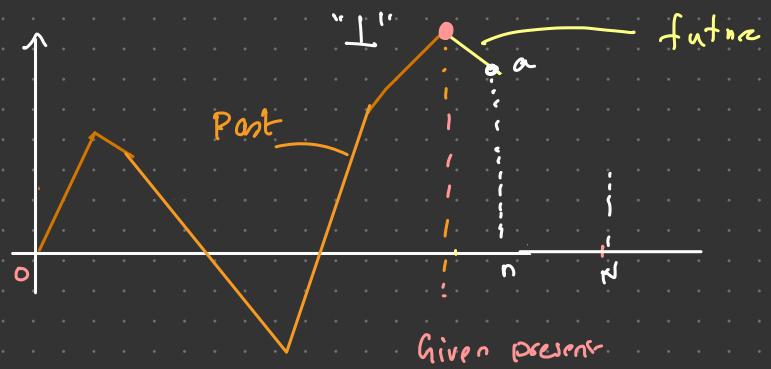
$\Rightarrow S_m - S_k = \sum_{n=k+1}^{m} X_n$

(+) $\mathbb{P}(S_n = a \mid S_{n-1} = a_{n-1}, \dots, S_1 = a_1)$ — (+)

$= \mathbb{P}(S_n = a \mid S_{n-1} = a_{n-1})$

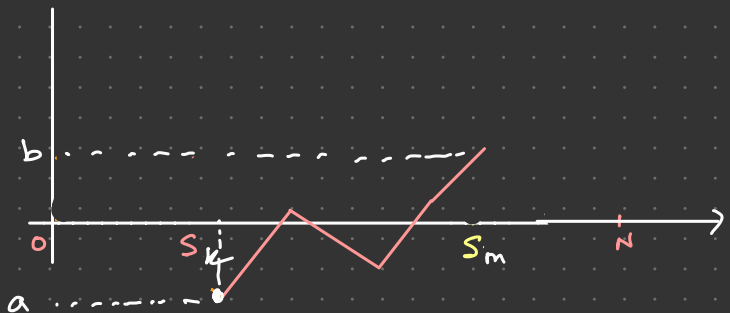
if $\mathbb{P}(S_{n-1} = a_{n-1}, \dots, S_1 = a_1) > 0$.

(+) (Markov Property)



(a) Suppose $\mathbb{P}(S_k = a) > 0$. for $k \geq 1$, $a \in \mathbb{Z}$.
$m > k$ $b \in \mathbb{Z}$

$$\mathbb{P}(S_m = b \mid S_k = a) = \mathbb{P}(S_{m-k} = b - a)$$



On an average where will the walk be :

Expectation $\mathbb{E}[S_n] = ?$ $\mathrm{Var}[S_n] = ?$

$$\mathbb{E}[X_k] = 1 \cdot \mathbb{P}(X_k = 1) + (-1)\,\mathbb{P}(X_k = -1)$$

$$= 1 \cdot \frac{1}{2} - \frac{1}{2} = 0 \quad - \#$$

$$S_n = \sum_{k=1}^{n} X_k + 0$$

$$\Rightarrow \mathbb{E}[S_n] = \sum_{k=1}^{n} \mathbb{E}[X_k] = 0$$

$$\mathrm{Var}[S_n] = \mathbb{E}[S_n^2] - (\mathbb{E}[S_n])^2$$

$$= \mathbb{E}[S_n^2] = \mathbb{E}\left(\left(\sum_{k=1}^{n} X_k\right)^2\right)$$

$$= E\left[\sum_{k=1}^{n} X_k^2 + \sum_{\substack{i,j \\ i \neq j}} X_i X_j\right]$$

$$= \sum_{k=1}^{n} E[X_k^2] + \sum_{\substack{i,j \\ i \neq j}} E[X_i X_j]$$

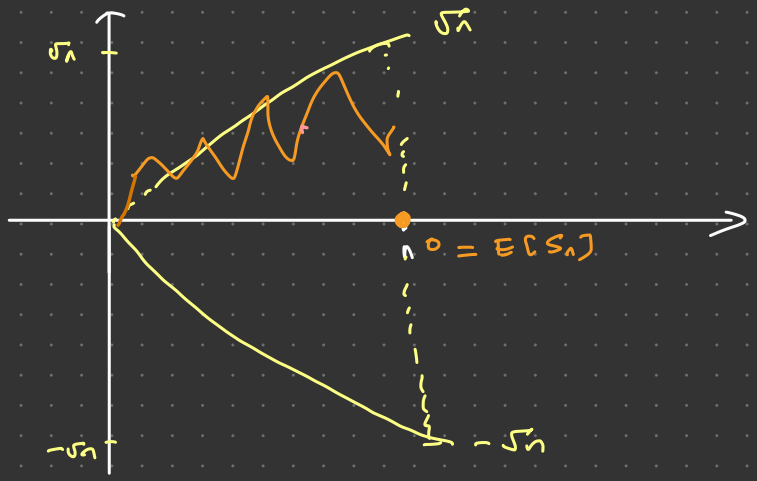
$$\left(\begin{matrix} i \neq j \\ X_i \perp X_j \\ \text{independent} \end{matrix}\right) = \sum_{k=1}^{n} E[X_k^2] + \sum_{\substack{i,j \\ i \neq j}} E[X_i]E[X_j]$$

(#)

Now $E[X_k^2] = 1^2 P(X_k = 1) + (-1)^2 P(X_k = -1)$

$= 1 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = 1$ $\quad \forall\, 1 \le k \le n$ (###)

$\Rightarrow$ $\mathrm{Var}[S_n] = n$, $\quad SD[S_n] = \sqrt{n}$



Lemma 1.1: $x \in \{-n, -n+2, \dots, n-2, n\}$

$$\mathbb{P}(S_n = x) = \binom{n}{\frac{n+x}{2}} \frac{1}{2^n}$$

Proof

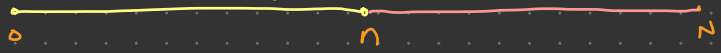
$\{S_n = x\} \overset{\text{Ex}}{\equiv}$ first n components of ω take precisely $\boxed{k = \frac{n+x}{2}}$ times value $+1$

i.e. $S_n = k(+1) + (n-k)(-1) = \boxed{2k - n \equiv x}$

k: # up/right steps; $(n-k)$: # down/left steps

# of elements ω s.t. $S_n(\omega) = x$

$:= |\{\omega \in \Omega \mid S_n(\omega) = x\}| = \binom{n}{k} 2^{N-n}$



we reach x in n steps

$$\mathbb{P}(S_n = x) = \frac{|\{\omega \in \Omega \mid S_n(\omega) = x\}|}{2^N}$$

$$= \frac{\binom{n}{k} 2^{N-n}}{2^N}$$

$$= \binom{n}{k} 2^{-n}$$

$$= \binom{n}{\frac{n+x}{2}} 2^{-n} \qquad \square$$

Observations

- Distribution of S_n - symmetric around 0.

$$\mathbb{P}(S_n = x) = \frac{n!}{\frac{n+x}{2}!\,\frac{n-x}{2}!} = \mathbb{P}(S_n = -x)$$

- (maximal weight) $\mathbb{P}(S_{2n} = 0) = \binom{2n}{n} 2^{-2n}$

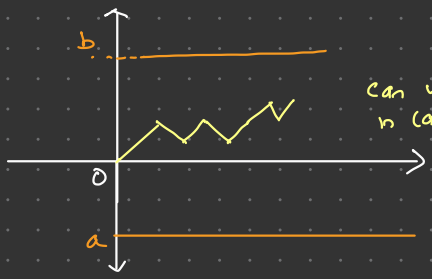
$$= \frac{2n!}{n!\,n!}\, 2^{-2n}$$

(Stirling's formula) $k! \underset{k\to\infty}{\sim} k^k e^{-k}\sqrt{2\pi k}$

$$\sim \frac{(2n)^{2n} e^{-2n} \sqrt{2\pi 2n} \cdot 2^{-2n}}{(n^n e^{-n}\sqrt{2\pi n})^2}$$

Ex. $= \frac{1}{\sqrt{\pi n}}$ (as $n \to \infty$)

Question



Can walk S_n stay in (a,b) forever?

$$0 \le \mathbb{P}(a \le S_n \le b) = \sum_{x \in [a,b]} \mathbb{P}(S_n = x)$$

$$\le \sum_{x \in [a,b]} [\mathbb{P}(S_n = 0) + \mathbb{P}(S_n = 0)]$$

for some $c_1 > 0$ $\leq (b-a+1)\frac{c_1}{\sqrt{n}}$ — (1)

By (1)

$\Rightarrow \mathbb{P}(a \leq S_n \leq b) \longrightarrow 0$ as $n \to \infty$ □

Mathematical issue:

- our $\mathbb{P} \equiv \mathbb{P}_N$ N - fixed. (length of walk)
- So $n \to \infty$ need $N \to \infty$ as well.
- Understanding $\mathbb{P}_N$ when $N = \infty$ [needs work] Come back to it later