Recall :-

- $(\Omega, \mathcal{F}, \mathbb{P})$ – on which $\{Z_n\}_{n \geq 1}$ is a Martingale if

  ① $E|Z_n| < \infty \quad \forall\, n \geq 1$

  ② $E[Z_n \mid Z_{n-1}, Z_{n-2}, \cdots, Z_1] = Z_{n-1} \quad \forall\, n \geq 2$

Interpretation :-

$$\begin{cases} f(z_1, z_2, \cdots, z_{n-1}) = E[Z_n \mid Z_{n-1} = z_{n-1}, \cdots, Z_1 = z_1] \\ \cdot\, E[Z_n \mid Z_{n-1}, Z_{n-2}, \cdots, Z_1] := f(Z_1, Z_2, \cdots, Z_{n-1}) \end{cases}$$

$E[Z_n \mid Z_{n-1} = z_{n-1}, \cdots, Z_1 = z_1] = z_{n-1}$; provided $\mathbb{P}(Z_{n-1} = z_{n-1}, \cdots, Z_1 = z_1) > 0$.

Random Walk

- $\{X_i\}_{i \geq 1}$ i.i.d. $X = \begin{cases} 1 & \text{w.p } \frac{1}{2} \\ -1 & \text{w.p } \frac{1}{2} \end{cases}$

$S_n = \sum_{i=1}^{n} X_i$ – $\{S_n\}_{n \geq 1}$ is a martingale

Product Martingale

- $\{X_i\}_{i \geq 1}$ i.i.d. X, with $E[X] = 1$

$Z_n = \prod_{i=1}^{n} X_i$ – $\{Z_n\}_{n \geq 1}$ is a martingale.

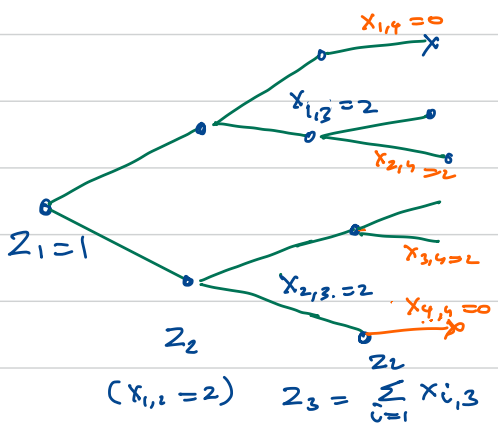
Example 3 (Branching – Process) – Critical

On $(\Omega, \mathcal{F}, \mathbb{P})$ suppose $\{X_{i,n}\}_{i,n \in \mathbb{N}}$ they are i.i.d. X such that

$$X = \begin{cases} 2 & \text{w.p. } \frac{1}{2} \\ 0 & \text{w.p. } \frac{1}{2} \end{cases}$$

Let $Z_1 = 1$, $Z_n = \sum_{i=1}^{Z_{n-1}} X_{i,n}$ $\forall\ n \geq 2$.

Think of $\{X_{i,n}\}$ $1 \leq i \leq Z_{n-1}$ as the offspring of each particle in Z_{n-1}.



$(X_{1,2}=2)$ $Z_3 = \sum_{i=1}^{Z_2} X_{i,3}$

(a realisation)

- Each particle alive in the nth generation gives birth to two or zero particle in the $n+1^{st}$ generation

- $E|Z_n| < \infty$

$n=1 \quad Z_1 = 1 \Rightarrow E|Z_1| < \infty$

Assume $n=k \quad E|Z_k| < \infty$

Then $n = k+1$

$$Z_{k+1} = \sum_{i=1}^{Z_k} X_{i,k+1}$$

$$E|Z_{k+1}| \leq 2\, E Z_k < \infty$$

(hypothesis)

$\Rightarrow$ by induction $E|Z_n| < \infty \quad \forall n \geq 1.$

- Assume $P(Z_{n-1} = \delta_{n-1}, \dots, Z_1 = \delta_1) > 0$

$$E[Z_n \mid Z_{n-1} = \delta_{n-1}, \dots, Z_1 = \delta_1]$$

$$= E\left[\sum_{i=1}^{Z_{n-1}} X_{i,n} \,\middle|\, Z_{n-1} = \delta_{n-1}, \dots, Z_1 = \delta_1\right]$$

$$= E\left[\sum_{i=1}^{\delta_{n-1}} X_{i,n} \,\middle|\, Z_{n-1} = \delta_{n-1}, \dots, Z_1 = \delta_1\right]$$

$$= \sum_{i=1}^{\delta_{n-1}} E\left[X_{i,n} \,\middle|\, Z_{n-1} = \delta_{n-1}, \dots, Z_1 = \delta_1\right]$$

As $E[X|Y] = E[X]$ when X is independent of Y this implies the above

$$= \sum_{i=1}^{Z_{n-1}} E[X_{i,n}]$$

$$= \sum_{i=1}^{Z_{n-1}} [2 \cdot \tfrac{1}{2} + 0 \cdot \tfrac{1}{2}] = Z_{n-1}$$

$\therefore$ $\{Z_n\}_{n \geq 1}$ is a Martingale

Ex: $\mathbb{P}(Z_n > 0) \longrightarrow 0$ as $n \to \infty$

Preliminaries

[Tower Property]

X, Y, Z are discrete random variables then

$$E[E[X|Y,Z] \mid Y] = E[X|Y]$$

Proof:- For $y \in \text{Range}(Y)$ $z \in \text{Range}(Z)$ let $\mathbb{P}(Y=y) > 0$, $\mathbb{P}(Z=z) > 0$, $\mathbb{P}(Y=y, Z=z) > 0$

$$h(y,z) := E[X|Y=y, Z=z]$$

$$= \sum_{x \in \text{Range}(X)} x \, \mathbb{P}(X=x \mid Y=y, Z=z)$$

Now, $E[E[X|Y,Z] \mid Y=y]$

$= E[h(Y,Z) \mid Y=y]$

$= \sum_{\substack{s \in \text{Range}(Y) \\ z \in \text{Range}(Z)}} h(s,z)\, P(Y=s, Z=z \mid Y=y)$

$= \sum_{z \in \text{Range}(Z)} h(y,z)\, P(Z=z \mid Y=y)$

$= \sum_{z \in \text{Range}(Z)} \left(\sum_{x \in \text{Range}(X)} x\, P(X=x \mid Y=y, Z=z) \right) P(Z=z \mid Y=y)$

(non-negative series)

$\overset{\bullet}{=} \sum_{x \in \text{Range}(X)} x \sum_{z \in \text{Range}(Z)} \frac{P(X=x, Y=y, Z=z)}{P(Y=y)}$

(all possibilities of z)

$\overset{\bullet}{=} \sum_{x \in \text{Range}(X)} x\, \frac{P(X=x, Y=y)}{P(Y=y)}$

$= E[X \mid Y=y]$

As $y \in \text{Range}(Y)$ and $z \in \text{Range}(Z)$ were arbitrary,

$$E[E[X|Y,Z]|Y] = E[X|Y] \qquad \square$$

2. Stopped Processes

On $(\Omega, \mathcal{F}, \mathbb{P})$ suppose X is a discrete random variable.

Definition 1:- (observable events)

$\mathcal{A}_X \equiv$ events observable by X.

$B \subseteq \Omega$ belongs to $\mathcal{A}_X$ if it can be written as a union of basic events

$$\{X = c\}, \quad c \in \text{Range}(X)$$

Definition 2: (observable events)

Suppose $Z_1, Z_2, \ldots, Z_n$ are discrete random variables for some $n \geq 1$

$\mathcal{A}_n =$ set of all observable events by $\{Z_i\}_{i=1}^n$

$B \subseteq \Omega$ belongs to $\mathcal{A}_n$ if it can be written as a union of basic events

$$\{Z_1 = z_1, \ldots, Z_n = z_n\}, \quad z_i \in \text{Range}(Z_i)$$

Sometimes $\mathcal{A}_n$ – referred to as set of all observable events by time n.

Next we shall define stopping times

Definition 3: (stopping time)

On $(\Omega, \mathcal{F}, \mathbb{P})$ let $\{Z_n\}_{n\geq 1}$ be a sequence of random variables.

$$T: \Omega \to \mathbb{N} \cup \{\infty\}$$

is called a stopping time for $\{Z_n\}_{n\geq 1}$ if

$\{T=k\}$ is an observable event from $\forall\, k \in \mathbb{N} \cup \{\infty\}$

$Z_1, \ldots Z_k$.

" $1_{\{T=k\}}$ is a function of $Z_1, Z_2 \ldots, Z_k$ "

i.e. $\{T=k\} \in \mathcal{A}_k$

Definition 4: (stopped process)

For any stopping time T of $\{Z_n\}_{n\geq 1}$ we define the stopped process

$$Z_n^T = \begin{cases} Z_n & n < T \\ Z_T & n \geq T \end{cases} \equiv Z_{n\wedge T}$$

$$Z_T \equiv Z_{T(\omega)}(\omega)$$

Theorem 2: Given a sequence $\{Z_n\}_{n\geq 1}$ and a stopping time $T : \Omega \to \mathbb{N} \cup \{\infty\}$ then the stopped process

$$\{Z_n^T\}_{n\geq 1} \text{ is a martingale}$$

whenever

$$\{Z_n\}_{n\geq 1} \text{ is a martingale.}$$

i.e.

- $E[Z_n^T] < \infty$
- $E[Z_n^T \mid Z_{n-1}^T, \dots, Z_1^T] = Z_{n-1}^T$ —(*)

when $\{Z_n\}_{n\geq 1}$ is a martingale.

[Intuitively —(*) is true:

$(z_{n-1}, \dots, z_1) \rightsquigarrow$ will determine if T has happened by time n

$\Downarrow$

$T \geq n \quad Z_n^T = Z_n$

$T < n \quad Z_n^T = Z_{n-1}$

$\{Z_n\}_{n\geq 1}$ - Martingale

$\Rightarrow E[Z_n^T \mid Z_{n-1}^T = z_{n-1}, \dots, Z_1^T = z_1] = z_{n-1}$]

Preliminaries on Conditional Expectation

Let $X_1, \dots, X_n$ & Z be discrete random variables

Suppose $\mathbb{P}(X_1 = x_1, \dots, X_n = x_n) > 0$.

- $E[Z \mid X_1 = x_1, \dots, X_n = x_n]$

$$= \sum_{k \in \text{Range}(Z)} k \, \mathbb{P}(Z = k \mid X_1 = x_1, \dots, X_n = x_n)$$

Now:

- $E[Z \mid X_1, X_2, \dots, X_n](\omega)$

$$= \sum_{x_i \in \text{Range}(X_i)} E[Z \mid X_1 = x_1, \dots, X_n = x_n] \cdot 1_{(X_1 = x_1 \dots X_n = x_n)}(\omega)$$

Suppose $\mathcal{B}_n \equiv$ set of all event observable by $X_1, X_2, \dots, X_n$

- $E[Z \mid \mathcal{B}_n] := E[Z \mid X_1 \dots, X_n]$

Example 4:- $\{X_i\}_{i \geq 1}$ i.i.d $\quad X = \begin{cases} 1 & \text{w.p } \frac{1}{2} \\ -1 & \text{w.p. } \frac{1}{2} \end{cases}$

$$S_n = \sum_{i=1}^{n} X_i$$

- $\xi_n = S_n^2 - n$ is a martingale

[Hw:- check ETS $\quad E[\xi_n \mid S_{n-1}, .., S_1] = \xi_{n-1}$]

$\Downarrow$?

Q: Is $E[\xi_n \mid \xi_{n-1}, \ldots, \xi_1] = \xi_{n-1}$?

clarification:-

$\mathcal{B}_n :=$ events observable by $\xi_1, \xi_2 \ldots, \xi_n$ $\quad | \; \xi_k = S_k^2 - k$

$\mathcal{A}_n :=$ events observable by $S_1, S_2, \ldots, S_n$ $\quad | \; S_k$

$\mathcal{B}_n \subseteq \mathcal{A}_n \quad \forall \; n \geq 1.$

$$E[\xi_n \mid \mathcal{B}_{n-1}]$$

$\mathcal{A}_{n-1} \supseteq \mathcal{B}_{n-1}$

$$\overset{\text{Tower Property}}{=} E\left[\, E[\xi_n \mid \mathcal{A}_{n-1}] \mid \mathcal{B}_{n-1}\right]$$

(xx)

$$= E[\xi_{n-1} \mid \mathcal{B}_{n-1}]$$

$$= E[\xi_{n-1} \mid \xi_{n-1}, \xi_{n-2}, \ldots \xi_1]$$

$E[x|x]=x$ ⟵

$$= \xi_{n-1}$$

Proof of Theorem 2 : To show

(i) $E\,|Z_n^T| < \infty$

(ii) $E[Z_n^T \mid Z_{n-1}^T, \ldots, Z_1^T] = Z_{n-1}^T$

Proof of (i):

Now: $$Z_n^T = \sum_{i=1}^{n-1} Z_i \, 1_{(T=i)} + Z_n^T \, 1_{(T \geq n)}$$

$$E\,|Z_n^T| = \sum_{i=1}^{n-1} E\left(|Z_i| \, 1_{(T=i)} + |Z_n| \, 1_{T \geq n}\right)$$

$$\leq \sum_{i=1}^{n} E|Z_n| < \infty$$

$(\because \{Z_n\}_{n\geq 1}$ is a martingale)

□

Proof of (ii):-

let $\mathcal{A}_n$ be observable events by $Z_1, \ldots, Z_n$

let $\mathcal{B}_n$ be observable events by $Z_1^T, \ldots, Z_n^T$

$$\mathcal{B}_n \subseteq \mathcal{A}_n.$$

$\therefore$ By Tower property trick (**) it is enough to show

$$E[Z_n^T \mid \mathcal{A}_{n-1}] = Z_{n-1}^T$$

Now

$$Z_n^T = \sum_{i=1}^{n-1} Z_i \, 1_{(T=i)} + Z_n \, 1_{(T\geq n)}$$

$$\therefore E[Z_n^T \mid \mathcal{A}_{n-1}]$$

linearity & conditional Expectation $= \sum_{i=1}^{n-1} E[Z_i \, 1_{(T=i)} \mid \mathcal{A}_{n-1}]$

$$+ E[Z_n \, 1_{(T\geq n)} \mid \mathcal{A}_{n-1}]$$

$$= \sum_{i=1}^{n-1} E[Z_i \, 1_{(T=i)} \mid Z_1, \dots, Z_{n-1}] + E[Z_n \, 1_{(T \geq n)} \mid Z_1 \dots Z_{n-1}]$$

$$\left(\begin{matrix} E[XY|Y] \\ = Y E[X|Y] \end{matrix} \right) = \sum_{i=1}^{n-1} Z_i \, 1_{(T=i)} + 1_{(T \geq n)} E[Z_n | Z_1 \dots Z_{n-1}]$$

$$\underset{(Z_n \text{ is martingale})}{=} \sum_{i=1}^{n-1} Z_i \, 1_{(T=i)} + 1_{(T \geq n)} Z_{n-1}$$

$$= \sum_{i=1}^{n-2} Z_i \, 1_{(T=i)} + 1_{(T \geq n-1)} Z_{n-1}$$

$$= Z_{n-1}^T$$

□

Corollary 3: T is a stopping time of a martingale $\{Z_n\}_{n \geq 1}$. Then

$$E[Z_n^T] = E[Z_1]$$

Proof: -

$\{Z_n^T\}_{n \geq 1}$ is a martingale

$$\Rightarrow E[Z_n^T] = E[Z_1^T] = E[Z_1]$$

□

Optional stopping Theorem :-

T is a stopping time of a martingale $\{Z_n\}_{n\geq 1}$

Ex: $Z_n^T \longrightarrow Z_T$ as $n \to \infty$.

Example 2(1) $\quad X = \begin{cases} 2 & \text{w.p } \frac{1}{2} \\ 0 & \text{w.p } \frac{1}{2} \end{cases}$

$Z_n = \prod_{i=1}^{n} X_i \qquad X_i$ are i.i.d X

- $P(Z_n = 0) = 1 - \frac{1}{2^n} \longrightarrow 1$ as $n \to \infty$

  $(Z_n \xrightarrow{p} 0$ as $n \to \infty)$

- $T = \min\{k \geq 1 \mid Z_k = 0\}$

  $Z_T = 0$ w.p. 1

  $\Rightarrow \quad E[Z_T] = 0$

- $E[Z_n] = E[Z_n^T] = E[Z_1] = 1$

- $Z_n^T \longrightarrow Z_T$ w.p. 1 as $n \to \infty$

  $\therefore \quad E[Z_T] \neq E[Z_1]$ □

Theorem 3:-

T is a stopping time of a martingale $\{Z_n\}_{n \geq 1}$ & $T: \Omega \to \mathbb{N}$.

$$E[Z_T] = E[Z_1] \iff$$

$$E[Z_n \mid T \geq n]\, P(T \geq n) \longrightarrow 0 \quad \text{as } n \to \infty.$$

& $E|Z_T| < \infty$

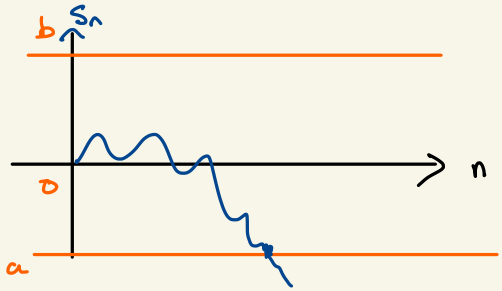
Application of OST :

Let $i \geq 1$ and $X_i = \begin{cases} 1 & \text{w.p } p \\ -1 & \text{w.p } 1-p \end{cases}$ be independent.

$0 < p < 1$

$S_n = \sum_{i=1}^{n} X_i \quad \forall\, n \geq 1$

$S_0 = 0$



$a < 0 < b$

$\sigma_a = \min \{ n \geq 1 \mid S_n = a \}$

$\sigma_b = \min \{ n \geq 1 \mid S_n = b \}$

Q: $P(\sigma_a < \sigma_b) = ?$

A:-

Ex:

- $T = \sigma_a \wedge \sigma_b$ then T is a stopping time
- $P(T \geq N) \longrightarrow 0$ as $N \to \infty$.

HW: $W_n = \left(\frac{1-p}{p}\right)^{S_n}$ is a martingale

Apply OST to $\{W_n\}_{n\geq 1}$. Need to show:

- $E[W_n | T \geq n]\, P(T \geq n) \longrightarrow 0$ as $n \to \infty$ - (*)
  and $E|W_T| < \infty$

Proof of (*):-

$$W_n = \left(\frac{1-p}{p}\right)^{S_n}$$

$$|W_n| \leq \left(\frac{1-p}{p}\right)^{a} + \left(\frac{1-p}{p}\right)^{b} \quad \text{if } T \geq n$$

$\Rightarrow$ (Ex) $\left|E(W_n | T \geq n)\right| \leq \left(\frac{1-p}{p}\right)^{a} + \left(\frac{1-p}{p}\right)^{b}$ - (†)

By Ex $P(T \geq n) \to 0$ as $n \to \infty$ (‡)

$\therefore$ (†) and (‡) $\Rightarrow$ (*)

$\therefore$ By (*) & OST we have

$$E[W_T] = E[W_1]$$

$$= \left(\frac{1-p}{p}\right) P(S_1 = 1) + \left(\frac{p}{1-p}\right) P(S_1 = -1)$$

$$= \left(\frac{1-p}{p}\right) p + \left(\frac{p}{1-p}\right)(1-p) = 1$$

□

But $E[W_T] = E(W_T ; \sigma_a < \sigma_b)$

$+ E(W_T ; \sigma_b \leq \sigma_a)$

$T = \sigma_a \wedge \sigma_b = E(W_{\sigma_a} ; \sigma_a < \sigma_b)$

$+ E(W_{\sigma_b} ; \sigma_b \leq \sigma_a)$

$S_{\sigma_a} = a$, $S_{\sigma_b} = b$ $= \left(\frac{1-p}{p}\right)^a \mathbb{P}(\sigma_a < \sigma_b)$

$+ \left(\frac{1-p}{p}\right)^b \mathbb{P}(\sigma_b \leq \sigma_a)$

$\Rightarrow \quad 1 = \left(\frac{1-p}{p}\right)^a \mathbb{P}(\sigma_a < \sigma_b) + \left(\frac{1-p}{p}\right)^b \mathbb{P}(\sigma_b \leq \sigma_a)$

$\Rightarrow \quad 1 = \left(\frac{1-p}{p}\right)^a \mathbb{P}(\sigma_a < \sigma_b) + \left(\frac{1-p}{p}\right)^b (1 - \mathbb{P}(\sigma_a < \sigma_b))$

$\Rightarrow$

$$\mathbb{P}(\sigma_a < \sigma_b) = \frac{\left(\frac{1-p}{p}\right)^b - 1}{\left(\frac{1-p}{p}\right)^b - \left(\frac{1-p}{p}\right)^a}$$

$a < 0 < b$

Proof of Theorem 3

- $\{Z_n^T\}_{n\geq 1}$ is a martingale

(*) $\Rightarrow E|Z_n^T| < \infty$ & $E[Z_n^T] = E[Z_1]$

$Z_n^T \longrightarrow Z_T$ as $n \to \infty$ w.p. 1

Now: $$E[Z_n^T] = \sum_{i=1}^{n-1} E[Z_n^T : T=i] + E[Z_n^T : T \geq n]$$

(Suitably interpreted if $P(T=i)=0$ for some i)

$$= \sum_{i=1}^{n-1} E[Z_n^T \mid T=i]\, P(T=i) + E[Z_n^T \mid T \geq n]\, P(T \geq n)$$

$$= \sum_{i=1}^{n-1} E[Z_i \mid T=i]\, P(T=i) + E[Z_n^T \mid T \geq n]\, P(T \geq n)$$

Using (*) and above

$$E[Z_1] = \sum_{i=1}^{n-1} E[Z_i \mid T=i]\, P(T=i) + E[Z_n \mid T \geq n]\, P(T \geq n) \quad \text{(**)}$$

Now as $E[Z_T] < \infty$,

$$E[Z_T] \overset{Ex}{=} \sum_{i=1}^{\infty} E[Z_T \mid T=i]\, P(T=i)$$

(Suitably interpreted if $P(T=i)=0$ for some i)

$$\overset{Ex}{=} \sum_{i=1}^{\infty} E[Z_i \mid T=i]\, P(T=i) \quad -(***)$$

Taking limits in (**) if

$$E[Z_n^* \mid T \geq n]\, P(T>n) \longrightarrow 0 \quad \text{as } n \to \infty$$

then $E[Z_T] = E[Z_1]$ ∎.

Checking assumptions of (OST):

- (Ex) If $T : \Omega \to \mathbb{N}$ and $\sup_{\omega \in \Omega} |T(\omega)| < M$

  then clearly assumption of (OST) holds

- (Ex) $P(T<\infty) = 1$ (i.e. $P(T>n) \to 0$ as $n \to \infty$)

  $(Z_n)_{n \geq 1}$ is a bounded martingale

  then clearly assumption of (OST) holds.

- (Hw) $E[T] < \infty$ and $\exists\, C > 0$ $|Z_n - Z_{n-1}| < C$ $\forall\, n \geq 2$

  then assumption of (OST) holds