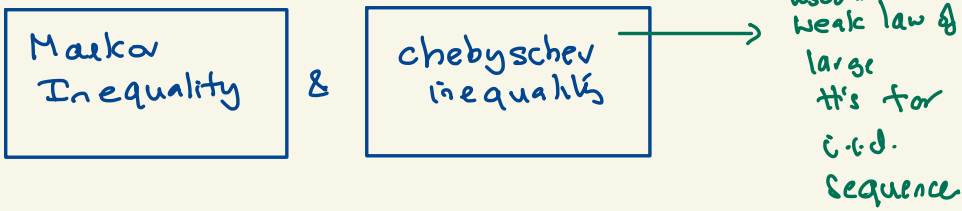
Kolmogorov Maximal Inequalities

These inequalities are more powerful than :



Kolmogorov Maximal Inequalities used in

- strong law of large numbers
- Martingale convergence theorem.
- Doob's inequality

Theorem 1: Let $\{Z_n\}_{n \geq 1}$ be a martingale such that $E[Z_n^2] < \infty \quad \forall\, n \geq 1$

Then $\forall\, m \geq 1$ and $b > 0$

$$P\left(\max_{1 \leq i \leq m} |Z_i| \geq b \right) \leq \frac{E[Z_m^2]}{b^2}$$

Preliminaries:

Suppose $h: \mathbb{R} \to \mathbb{R}$ is a convex function and $\{Z_n\}_{n \geq 1}$ is a martingale & $E|h(Z_n)| < \infty$ $\forall n \geq 1$

then $E[h(Z_n) \mid Z_{n-1}, .. Z_1]$

$\overset{\text{(Jensen)}}{\geq} h(E(Z_n \mid Z_{n-1}, \cdot, Z_1))$

$= h(Z_{n-1})$

we say $\{h(Z_n)\}_{n \geq 1}$ is a sub-martingale!

Definition: $\{Z_n\}_{n \geq 1}$ a sequence of random variables is said to be a sub-martingale if $E|Z_n| < \infty$ and

$E[Z_n \mid Z_{n-1}, Z_{n-2}, \cdots, Z_1] \geq Z_{n-1}$ $\forall n \geq 2$.

Example: $\{Z_n\}_{n \geq 1}$ be a martingale

let $h(x) = x^2$ and $EZ_n^2 < \infty$

$\Rightarrow$ $\{h(Z_n)\}_{n \geq 1}$ is a sub-martingale

let $h(x) = e^{\alpha x}$ and $E e^{\alpha Z_n} < \infty$

$\Rightarrow \{h(Z_n)\}_{n \geq 1}$ is a sub-martingale

Theorem 1' (Kolmogorov inequality)

Let $\{Z_n\}_{n \geq 1}$ be a non-negative submartingale such

Then $\forall\, m \geq 1$ and $a > 0$

$$P\left(\max_{1 \leq i \leq m} Z_i \geq a \right) \leq \frac{E[Z_m]}{a}$$

Proof of Theorem 1

Observe if $\{Z_n\}_{n \geq 1}$ is a martingale $\Rightarrow \{Z_n^2\}_{n \geq 1}$ is a sub martingale

Then take $a = b^2$ in Theorem 1' to $\{Z_n^2\}_{n \geq 1}$ obtain the result $\square$

Proof of Theorem 1' :-

We need to understand $\rightarrow P\left(\max_{1 \leq i \leq m} Z_i \geq a \right)$

let $T = \min\{k \geq 1 \mid Z_k \geq a\} \wedge m$

If $Z_n < a \quad \forall \; n < m \implies J = m$!

$\overset{(Ex)}{\implies} \quad Z_J \geq a$ iff $Z_n \geq a$ for some $n \leq m$

$$\therefore \quad \mathbb{P}\left(\max_{1 \leq i \leq n} Z_i \geq a \right) = \mathbb{P}\left(Z_J \geq a \right)$$

Markov Inequality

$$\leq \frac{E[Z_J]}{a} \quad - (\star)$$

$$E[Z_J] = E\left[\sum_{k=1}^{m-1} Z_k \mathbb{1}_{J=k} + Z_m \mathbb{1}_{J \geq m} \right]$$

$$= \sum_{k=1}^{m-1} E[Z_k \mathbb{1}_{J=k}] + E[Z_m \mathbb{1}_{J \geq m}]$$

A_k - observable by events $Z_1, \ldots, Z_k$ & $E[Z_m | A_m] = Z_k$

$$= \sum_{k=1}^{m-1} E\left[E[Z_m | A_k] \mathbb{1}_{J=k} \right] + E[Z_m \mathbb{1}_{J \geq m}]$$

$E[XY|A_Y] = Y E[X|A_Y]$

$$= \sum_{k=1}^{m-1} E\left[E[Z_m \mathbb{1}_{J=k} | A_k] \right] + E[Z_m \mathbb{1}_{J \geq m}]$$

- J is stopping time

$$E[\,E[X|Y]\,] = E[X]$$

$$= \sum_{k=1}^{m-1} E[\,Z_m \mathbb{1}_{J=k}\,] + E[Z_m \mathbb{1}_{J\geq m}]$$

$$= E\left[Z_m \left[\sum_{k=1}^{m-1} \mathbb{1}_{J=k} + \mathbb{1}_{J\geq m}\right]\right]$$

$$= E[Z_m] \quad - (*)$$

$(\circledast)$ in $(*)$ $\Rightarrow$

$$\mathbb{P}\left(\max_{1\leq i\leq m} Z_i \geq a\right) \leq \frac{E\,Z_m}{a}$$

□

Theorem 2: Let $\{Z_n\}_{n\geq 1}$ be a non-negative martingale. Then $\forall a > 0$

$$\mathbb{P}\left(\sup_{n\geq 1} Z_n \geq a\right) \leq \frac{E[Z_1]}{a}$$

Proof:- In particular $\{Z_n\}_{n\geq 1}$ is a non-negative submartingale.

$\forall\, m \geq 1$ $\quad \mathbb{P}\left(\max_{1\leq i\leq m} Z_i \geq a\right) \leq \frac{E\,Z_m}{a} = \frac{E\,Z_1}{a}$

↑ Martingale

If $A_m = \left\{\max_{1\leq i\leq m} Z_i \geq a\right\}$ $\Rightarrow$ $A_m \subseteq A_{m+1}$ $\forall\, m \geq 1$

$$A_m \uparrow A = \bigcup_{i \geq 1} \{Z_i \geq a\} \quad \text{(key)}$$

let $\varepsilon_0 > 0$ be given & so that $0 < \varepsilon_0 < a$

$$P\left(\sup_{n \geq 1} Z_n \geq a\right) = P\left(\bigcap_{\varepsilon > 0} \bigcup_{i=1}^{\infty} \{Z_i \geq a - \varepsilon\}\right)$$

$$\leq P\left(\bigcup_{i=1}^{\infty} \{Z_i \geq a - \varepsilon_0\}\right) \quad - (*)$$

Apply Theorem 1' to $b = a - \varepsilon_0$ to get

$$P\left(\max_{1 \leq i \leq m} Z_i \geq a - \varepsilon_0\right) \leq \frac{E[Z_1]}{a - \varepsilon_0} \quad - (+)$$

$$B_m^{\varepsilon_0} = \left\{\max_{1 \leq i \leq m} Z_i \geq a - \varepsilon_0\right\} \uparrow \bigcup_{i \geq 1} \{Z_i \geq a - \varepsilon_0\} \quad (++)$$

$\therefore$ (+) and (++)

$$P\left(\bigcup_{i \geq 1} \{Z_i \geq a - \varepsilon_0\}\right) \leq \frac{E Z_1}{a - \varepsilon_0}$$

From (*)

$$P\left(\sup_{n \geq 1} Z_n \geq a\right) \leq \frac{E Z_1}{a - \varepsilon_0} \quad \forall \ 0 < \varepsilon_0 < a$$

Take $\varepsilon_0 \to 0$ on R.H.S to get

$$\mathbb{P}\left(\sup_{n \geq 1} Z_n \geq a\right) \leq \frac{\mathbb{E} Z_1}{a}$$

D

[Martingale Convergence Theorem]

Let $\{Z_n\}_{n\geq 1}$ be a martingale and assume that

$\exists\ M>0$ such that $E[Z_n^2] \leq M,\ \forall\ n\geq 1$

Then $\exists$ a r.v. Z such that

$$P\left(Z_n \longrightarrow Z \text{ as } n\to\infty\right) = 1$$

Proof:

- • $\{Z_n^2\}$ is a sub-martingale.

(Ex.) $\Rightarrow$ $E[Z_n^2 \mid Z_{n-1}, \dots Z_1] \geq Z_{n-1}^2 \quad \forall n \geq 2$

Ex. Take Expectation abov

$\Rightarrow$ $E Z_n^2 \geq E Z_{n-1}^2 \quad \forall n \geq 2$

$\therefore$ By $E Z_n^2 \leq n \quad \forall n \geq 1 \Rightarrow \exists \alpha \geq 0$ Such that

$E Z_n^2 \longrightarrow \alpha$ as $n \to \infty$ — (*)

Hw: let $m \geq 1$ & $Y_n = Z_{n+m} - Z_m \quad \forall n \geq 1$

then $\{Y_n\}_{n \geq 1}$ is a martingale.

By kolmogorov maximal inequality $\forall n \geq 2, \forall b > 0$

$$P\left(\max_{1 \leq i \leq n} |Y_i| \geq b\right) \leq \frac{E Y_n^2}{b^2}$$ — (+)

provided $E Y_n^2 < \infty \quad \forall n \geq 1$

Proof of $EY_n^2 < \infty$:

$$EY_n^2 = E\left[(Z_{n+m} - Z_m)^2\right]$$

$$= E Z_{n+m}^2 + E Z_m^2 - 2 E\left[Z_{n+m} Z_m\right]$$

$$= E Z_{n+m}^2 + E Z_m^2 - 2 E\left[E\left[Z_{n+m} Z_m \mid A_m\right]\right]$$

where A_m is the observable events of $Z_1, \ldots, Z_m$

$E[XY|A_X] = X E[Y|A_X]$

$$= E Z_{n+m}^2 + E Z_m^2 - 2 E\left[Z_m E\left[Z_{n+m} \mid A_m\right]\right]$$

$\{Z_n\}_{n\geq 1}$ is a martingale

$$= E Z_{n+m}^2 + E Z_m^2 - 2 E Z_m^2$$

$$= E Z_{n+m}^2 - E Z_m^2 < \infty.$$

□

By (+)

$$\Rightarrow P\left(\max_{1 \leq i \leq n} |Z_{i+m} - Z_m| \geq b\right) \leq \frac{E(Z_{n+m} - Z_m)^2}{b^2}$$

$$\Rightarrow \quad \mathbb{P}\left(\max_{1 \le i \le n} |Z_{i+m} - Z_m| \ge b \right) \le \frac{\mathbb{E} Z_{n+m}^2 - \mathbb{E} Z_m^2}{b^2}$$

let $n \to \infty$ on both sides

$\mathbb{E} Z_{n+m}^2 \to \alpha$ as $n \to \infty$

$$0 \le \mathbb{P}\left(\bigcup_{i \ge 1} |Z_{i+m} - Z_m| \ge b \right) \le \frac{\alpha - \mathbb{E} Z_m^2}{b^2}$$

Take $m \to \infty$ on both sides ($\mathbb{E} Z_m^2 \to \alpha$ as $m \to \infty$)

$\forall\, b > 0$

$$\lim_{m \to \infty} \mathbb{P}\left(\bigcup_{i \ge 1} |Z_{i+m} - Z_m| \ge b \right) = 0$$

$+ +$

Claim :- $\mathbb{P}(\{Z_m\}_{m \ge 1}$ is Cauchy sequence $) = 1$

Proof :-

$$A = \{ \omega \in \Omega \mid \forall\, b > 0 \ \exists\, N \ge 1 \ |Z_{i+m} - Z_m| < b, \ \forall\, m \ge N \ \& \ i \ge 1 \}$$

To show $\mathbb{P}(A) = 1$.

$$A = \bigcap_{b > 0} \bigcup_{N \ge 1} \bigcap_{m=N}^{\infty} \bigcap_{i=1}^{\infty} \{ |Z_{i+m} - Z_m| < b \}$$

Let $\ell \geq 1$ $\quad b = \frac{1}{2^\ell}$ by (++) $\exists\, k_\ell \geq 1$

$$\mathbb{P}\left(\sup_{i \geq 1} |z_{i+k_\ell} - z_{k_\ell}| > \frac{1}{2^\ell} \right) \leq \frac{1}{2^\ell}$$

$$\Rightarrow \sum_{\ell=1}^{\infty} \mathbb{P}\left(\sup_{i \geq 1} |z_{i+k_\ell} - z_{k_\ell}| > \frac{1}{2^\ell} \right) < \infty$$

By Borel-Cantelli Lemma, if $A_\ell = \left\{ \sup_{i \geq 1} |z_{i+k_\ell} - z_{k_\ell}| > \frac{1}{2^\ell} \right\}$

$$\Rightarrow \mathbb{P}\left(\bigcap_{n \geq 1} \bigcup_{m=n}^{\infty} A_m \right) = 0$$

$$\Rightarrow \mathbb{P}\left(\{ w \in \Omega \mid w \in A_\ell \text{ for infinitely many } \ell \geq 1 \} \right) = 0$$

$$\Rightarrow \mathbb{P}\left(\{ w \in \Omega \mid w \in A_\ell \text{ for finitely many } \ell \geq 1 \} \right) = 1$$

$$\Rightarrow \mathbb{P}\left(\{ w \in \Omega \mid \exists\, \ell_0 \geq 1 \ \forall\, \ell \geq \ell_0, \ \sup_{i \geq 1} |z_{i+k_\ell} - z_{k_\ell}| \leq \frac{1}{2^\ell} \} \right) = 1$$

$\Downarrow$ Ex

$$\Rightarrow \mathbb{P}\left(\bigcap_{b > 0} \bigcup_{N \geq 1} \bigcap_{m=N}^{\infty} \bigcap_{i=1}^{\infty} \{ |z_{i+m} - z_i| < b \} \right) = 1$$

$\Rightarrow \quad \mathbb{P}(\{Z_k\}_{k\geq 1} \text{ is Cauchy sequence}) = 1$

$\Rightarrow \mathbb{P}(\{\omega \in \Omega \mid Z_k(\omega) \longrightarrow Z(\omega) \text{ as } k \to \infty\}) = 1$

Ex. Z is r.v and $\mathbb{P}(Z < \infty) = 1$. □

[Martingale Convergence Theorem 2]

Suppose $\{Z_n\}_{n\geq 1}$ is a martingale and

$$\sup_{n\geq 1} \mathbb{E}|Z_n| < \infty$$ then

$\exists$ a r.v. Z such that

$$\mathbb{P}(Z < \infty) \quad \& \quad \mathbb{P}(Z_n \to Z \text{ as } n \to \infty) = 1.$$

Corollary:

Suppose $\{Z_n\}_{n\geq 1}$ is a non-negative martingale then

$\exists$ a r.v. Z such that

$$\mathbb{P}(Z < \infty) \quad \& \quad \mathbb{P}(Z_n \to Z \text{ as } n \to \infty) = 1.$$

Proof of Corollary: $Z_n \geq 0 \Rightarrow$

$\{Z_n\}_{n\geq 1}$ is a martingale

$$E|Z_n| = EZ_n = EZ_1 < \infty$$

By [Martingale Convergence Theorem 2]

we have the result $\square$

Proof of Martingale Convergence Theorem 2 :

Intuition :-

$x_n \in \mathbb{R}$

$x_n \to x$ as $n \to \infty$

$x_n \to \infty$ as $n \to \infty$

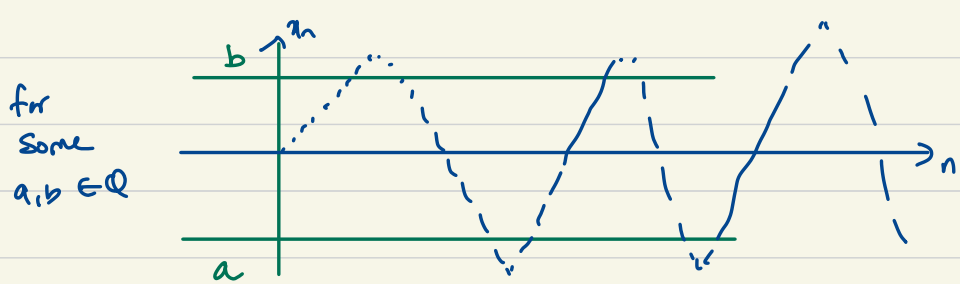
$x_n \to -\infty$ as $n \to \infty$

$$\overline{\lim_{n\to\infty}} x_n = \underline{\lim_{n\to\infty}} x_n$$

x_n does not converge to $a \in \mathbb{R} \cup \{\infty\} \cup \{-\infty\}$

$$\underline{\lim_{n\to\infty}} x_n < \overline{\lim_{n\to\infty}} x_n$$

x_n does not converge to $a \in \mathbb{R} \cup \{\infty\} \cup \{-\infty\}$ $\equiv \bigcup_{a,b \in \mathbb{Q}} \left\{ \underline{\lim_{n\to\infty}} x_n < a < b < \overline{\lim_{n\to\infty}} x_n \right\}$



Crossings of (a,b) is infinite

let $\{Z_n\}_{n\geq 1}$ be a martingale. For $a,b \in \mathbb{Q}$

$$\Lambda_{a,b} = \left\{ \omega \in \Omega \;\middle|\; \underline{\lim_{n\to\infty}} Z_n(\omega) < a < b < \overline{\lim_{n\to\infty}} Z_n(\omega) \right\}$$

$$\mathbb{P}\left(\{Z_n\}_{n\geq 1} \text{ does not converge to } a \in \mathbb{R} \cup \{\pm\infty\} \right)$$

$$= \mathbb{P}\left(\bigcup_{a,b\in\mathbb{Q}} \Lambda_{a,b} \right)$$

$$\underset{Ex}{\leq} \sum_{a,b\in\mathbb{Q}} \mathbb{P}(\Lambda_{a,b})$$

We will first show $\mathbb{P}(\Lambda_{a,b}) = 0 \quad \forall a,b \in \mathbb{Q}$.

$S_0 = 0$

let $n \geq 1$ $\quad S_1 = \min\{n \geq 1 \mid Z_n < a\}$

$t_n = \min\{n \geq S_{n-1} \mid Z_n \geq b\}$

$n \geq 2$: $\quad S_n = \min\{n \geq t_{n-1} \mid Z_n < a\}$

For $N \geq 1$

$U_N(a,b) \equiv \max\{k \geq 1 \mid 1 \leq S_1 \leq t_1 \leq \dots < t_k \leq N\}$

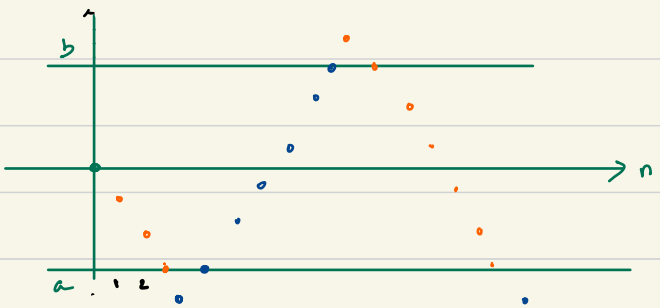
# of crossing of (a,b) until time N.

$N \to \infty \qquad U_N(a,b) \longrightarrow U(a,b)$

$U(a,b)$ = # of (crossings) of (a,b) by $\{Z_n\}_{n\geq 1}$

$\therefore \quad \Lambda_{a,b} = \{ U(a,b) = \infty \}.$

E.T.S. $\quad \mathbb{P}(U(a,b) = \infty) = 0$



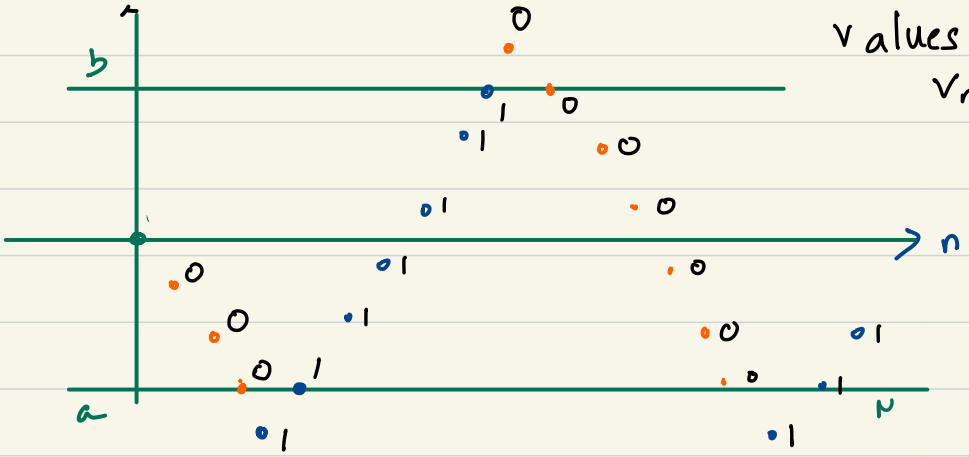
values of Z_n

$V_1 = 0 \; ; \quad V_2 = 1(Z_1 < a)$

$V_n = 1(V_{n-1}=1, Z_{n-1} \leq b) + 1(V_{n-1}=0, Z_{n-1} < a)$

when 2 enters the region below a V takes the value 1 & will keep assuming 1 till 2 enters the region above b. Then V takes value 0 & will keep assuming 0 till 2 enters the region below a.

Fix $N \geq 2 \qquad Z_N^V = \sum_{k=2}^{N} V_k (Z_k - Z_{k-1})$



(Ex) $z_N^v \geq (b-a)\, u_N(a,b) - (z_N - a)^-$

and

$$E[z_N^v] = E\left[\sum_{k=2}^{N} v_k (z_k - z_{k-1})\right]$$

(A_n - events observable by $z_1, \ldots, z_n$)

$$= \sum_{k=2}^{N} E\left[E\left[v_k (z_k - z_{k-1}) \,\middle|\, A_{k-1}\right]\right]$$

(v_k - is observable by events in A_{k-1})

$$= \sum_{k=2}^{N} E\, v_k \left[E[z_k | A_{k-1}] - z_{k-1}\right]$$

(z_n is a martingale)

$$= \sum_{k=2}^{N} E\, v_k \cdot 0$$

$$= 0$$

$$\Rightarrow \quad U_N(a,b) \le \frac{1}{b-a}\left(Z_N^{\nu} + (Z_N - a)^{-}\right)$$

$$\Rightarrow \quad E[U_N(a,b)) \le \frac{1}{b-a}\left(E Z_N^{\nu} + E(Z_N - a)^{-}\right)$$

$\{Z_n\}_{n\ge 1}$ martingale and $(Z_n - a)^{-} < |Z_n| + |a|$

$$\le \frac{1}{b-a}\left(0 + E|Z_n| + |a|\right)$$

$$\le C \quad (\text{say}) \qquad (\ddagger)$$

- $0 \le U_N(a,b) \uparrow U(a,b)$ as $N \to \infty$

(Ex). $E(U_N(a,b)) \uparrow E\,U(a,b)$

Using this and $(\ddagger)$ we have

$$E\,U(a,b) \le C$$

$$\Rightarrow \quad P(U(a,b) < \infty) = 1$$

$$\Rightarrow \quad P(U(a,b) = \infty) = 0$$

$\Rightarrow \quad P(\ Z_n \to Z \text{ as } n \to \infty \quad Z \in \mathbb{R} \cup \{\infty\} \cup \{-\infty\}) = 1$

(Ex) Z is r.v

Consider $E|Z| = E \lim_{n\to\infty} |Z_n|$

$\overset{\text{Fatou's lemma}}{\leq} \lim_{n\to\infty} E|Z_n|$

$\leq \sup_{n\geq 1} E|Z_n| < \infty$ (Assumption)

$\Rightarrow P(|Z| < \infty) = 1$

$\Rightarrow \quad P(\ Z_n \to Z \text{ as } n \to \infty \quad Z \in \mathbb{R}) = 1$ $\square$