Models

Q:- Donkey randomly presses keyboard. What is the chance that "ISP IS BORING" will appear ?

Let $(\psi_1, \psi_2, \ldots \psi_{10}, \psi_{11}) = (I, S, P, I, S, B, O, R, I, N, G)$

Q: $\{U_i\}_{i \geq 1}$ are i.i.d Uniform(S) where $S = \{A, B, \ldots, Z\}$

$\tau = \min\{n \geq 11 \mid (U_{n-10}, U_{n-9}, \ldots U_n) = (\text{ISP IS BORING})\}$

What is $E[\tau] = ?$

A: (Markov chain)

$Y_1 = (U_1, U_2, \ldots, U_{11})$

$Y_2 = (U_2, \ldots, U_{12})$

$\vdots$

$Y_k = (U_k, \ldots, U_{k+10})$

$Y_k \in S^{11}$

Set $Y_0 = (A\,B\,C\,D\,E\,F\,G\,H\,I\,J\,K)$

Ex: $\{Y_k\}_{k \geq 1}$ is a Markov chain on S^{11}

$P \equiv$ can be identified.

Find $E_{Y_0}[\tau] = ?$

Set $h(y) = E_y(\tau)$

.... Ex.. h is a minimal solution to ...

$$h(I,S,P,I,S,B,O,R,I,N,G) = 0$$

and

$$h(y) = \sum_{z \in S} P_{yz} h(z)$$

$$y \neq (I,S,P,I,S,B,O,R,I,N,G)$$

Solving ... By hand is hard. □

Answer 2 : Use Martingales - ISPIS BORING

Casino :



$\{Z^1_k\}_{k\geq 1}$ = capital of gambler arrives at $t=1$

$\boxed{Z^1_0 = 1}$ $t=1$

places a bet 1 Rs & chooses a letter

- If letter = I then gambler gets 26 Rs
  If letter $\neq$ I then gambler gets 0 Rs and exits casino.

Expected Profit : $\frac{1}{26}(26-1) + \frac{25}{26}(-1) = 0$

$t=2$ If $Z^1_1 = 26$ & Gambler bets all 26 Rs.

- If letter = S then gambler gets $(26)^2$ Rs
  If letter $\neq$ S then gambler gets 0 Rs and exits casino.

and so on till $t=11$ & $Z^1_t = Z^1_{11}$ $\forall\, t \geq 10$

For any $k \geq 1$, let gambler k enter the casino at $t=k$

let $Z^k_n = \begin{cases} 1 & n < k \\ 26\, Z^k_{n-1}\, \mathbb{1}_{(U_n = \Psi_{n-k+1})} & k \leq n \leq k+10 \\ Z^k_{k+10} & n \geq k+11 \end{cases}$

Net Profit of casino from k^{th} Gamble =

$$= 1 - Z_n^k$$

Let $M_n = \sum_{k=1}^{n} (1 - Z_n^k) = n - \sum_{k=1}^{n} Z_n^k$

Claim 1: (i) $\{M_n\}_{n \geq 1}$ is a martingale

(ii) τ - stopping time & OST hypothesis are satisfied

Proof:

(i) $E[M_n \mid U_{n-1}, \dots, U_1]$

$$= E\left[n - \sum_{k=1}^{n} Z_n^k \;\middle|\; U_{n-1}, \dots, U_1\right]$$

$$= n - E\left[\sum_{k=1}^{n-11} Z_n^k + \sum_{k=n-10}^{n} Z_k^n \;\middle|\; U_{n-1}, \dots, U_1\right]$$

By definition $= n - E\left[\sum_{k=1}^{n-11} Z_{n-1}^k + \sum_{k=n-10}^{n} 26\, Z_{n-1}^k \mathbb{1}_{(U_n = \Psi_{n-k+1})} \;\middle|\; U_{n-1}, \dots, U_1\right]$

U_n is independent of $U_{n-1}, \dots, U_1$ $= n - \left(\sum_{k=1}^{n-11} Z_{n-1}^k + \sum_{k=n-10}^{n} 26\, Z_{n-1}^k\, \mathbb{P}(U_n = \Psi_{n-k+1})\right)$

Ex $= n - 1 - \sum_{k=1}^{n-1} Z_{n-1}^k$

$\Rightarrow \{M_n\}_{n \geq 1}$ is a Martingale.

(ii)

$$\{\tau = k\} = \{(u_{k-10}, u_{k-9}, \ldots, u_k) = (I,S,P,I,S,B,O,R,I,N,G)\}$$

$$\bigcap_{j=11}^{k-9} \{(u_{j-10}, u_{j-9}, \ldots, u_j) \neq (I,S,P,I,S,B,O,R,I,N,G)\}$$

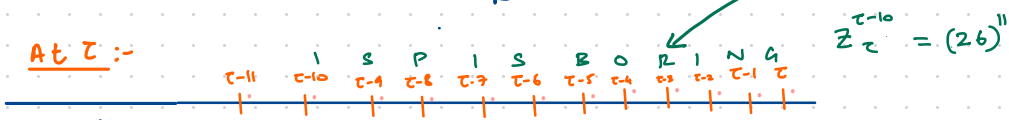
$\therefore \{\tau = k\}$ is observable from $u_1, \ldots, u_k$.

So its a stopping time.

$M_1 = 0$ &

Now $M_n = n - \sum_{k=1}^{n} Z_n^k$

$M_\tau = \tau - \sum_{k=1}^{\tau} Z_\tau^k$



As there is no repetition of initial pair of string

$Z_\tau^k = 0$ for $\tau-10 < k \leq \tau$ &

$\Rightarrow M_\tau = \tau - (26)''$

Now for $n \geq 2$,

$$|M_n - M_{n-1}| = \left| \sum_{k=1}^{n-1} (Z_{n-1}^k - Z_n^k) - Z_n^n + 1 \right|$$

if $n > k+10$

$Z_n^k - Z_{n-1}^k = 0$

and $Z_n^n \leq 26$

$$\leq \left| \sum_{k=n-10}^{n-1} (Z_{n-1}^k - Z_n^k) \right| + 25$$

$$= \left| \sum_{k=n-10}^{n-1} \left(26\, 1(u_n = \psi_{n-k+1}) - 1\right) Z_{n-1}^k \right| + 25$$

$$\leq 25 \sum_{k=n-10}^{n-1} |Z_{n-1}^k| + 25$$

$Z_k^1 \leq 25^{11}$

$$\leq (25)^{12} \left(n-1-(n-10)+1 \right) + 25$$

$$\leq (25)^{12} \cdot 10 + 25 := c \quad - (\mp)$$

Let for $n \geq 1$, divide the sequence

$u_1\ u_2\ \underbrace{u_{11}\ \ldots\ \ldots}_{w_1}\ \underbrace{\ldots\ u_k\ \ldots\ \ldots\ u_n}_{w_2}\ \ldots$

into blocks of size 11.

Let S be the number of blocks until a block $\equiv (\psi_1, \ldots, \psi_{11})$

Then $S \sim$ Geometric $\left(\frac{1}{26^{11}}\right)$

and $\tau \leq 11 S$ (Ex.)

$\Rightarrow$ $E\tau < 11\, E[S] = 11(26)^{11} < \infty$

$- (+)$

$\therefore$ By $(+)$ and $(\mp)$

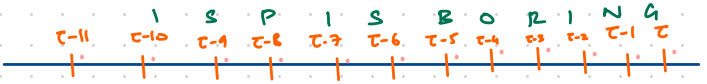
OST hypothesis are satisfied

By OST : $E[M_\tau] = E[M_1]$ — (*)

$$E[M_1] = E(1 - Z_1^1) = 1 - \frac{26}{26} + \frac{25}{26}(0) = 0$$

Now $$M_n = n - \sum_{k=1}^{n} Z_n^k$$

$$M_\tau = \tau - \sum_{k=1}^{\tau} Z_\tau^k$$



$Z_\tau^k = 0$ for $\tau - 10 < k \leq \tau$ &

$$Z_\tau^{\tau-10} = (26)^{11}$$

$$M_\tau = \tau - (26)^{11}$$

$$E[M_\tau] = 0 \Rightarrow E[\tau] = (26)^{11}$$ □