Erdös-Renyi $(n, \frac{\lambda}{n})$, $\lambda < n$. , $\lambda \equiv \lambda_n$

- Vertex set $\{1, \dots, n\}$
- Edge set $i \sim j$ w.p $\frac{\lambda}{n}$ (independently)

Q: Probability that Erdös-Renyi $(n, \frac{\lambda}{n})$, is connected ?

Claim 1:- Set Y_n = # of isolated vertices

(a) $n e^{-\lambda} e^{-\lambda^2/n} \leq E[Y_n] \leq n e^{-\lambda} e^{\lambda/n}$

(b) $\text{var}[Y_n] \leq E[Y_n] + \frac{\lambda}{n-\lambda} (E[Y_n])^2$

Proof:- (a) $E[Y_n] = n(1 - \frac{\lambda}{n})^{n-1}$

Use $1 - x \geq e^{-x-x^2}$ and $1 - x \leq e^{-x}$
$0 \leq x \leq \frac{1}{2}$

(b) $E[Y_n^2] \overset{\text{Exp}}{=} n \, \mathbb{P}\left(1(\text{vertex } 1 \text{ is a isolated vertex in } G(n, \frac{\lambda}{n}))\right)$
$+ n(n-1) \, \mathbb{P}\left(1(\text{vertex } 1 \text{ is a isolated vertex in } G(n, \frac{\lambda}{n})) \; 1(\text{vertex } 2 \text{ is a isolated vertex in } G(n, \frac{\lambda}{n}))\right)$

$\text{var}[Y_n] = E[Y_n^2] - (E[Y_n])^2$

$= E[Y_n^2] - \left(n \, \mathbb{P}\left(1(\text{vertex } 1 \text{ is a isolated vertex in } G(n, \frac{\lambda}{n}))\right)\right)^2$

$= n \, \mathbb{P}\left(1(\text{vertex } 1 \text{ is a isolated vertex in } G(n, \frac{\lambda}{n}))\right)$
$+ n(n-1) \, \mathbb{P}\left(1(\text{vertex } 1 \text{ is a isolated vertex in } G(n, \frac{\lambda}{n})) \; 1(\text{vertex } 2 \text{ is a isolated vertex in } G(n, \frac{\lambda}{n}))\right)$
$- n^2 \left(\mathbb{P}\left(1(\text{vertex } 1 \text{ is a isolated vertex in } G(n, \frac{\lambda}{n}))\right)\right)^2$

□

$$= n\left[\mathbb{P}\left(1(\text{vertex 1 is a isolated vertex in } G(n,\tfrac{\lambda}{n}))\right) - \mathbb{P}\left(1(\text{vertex 1 is a isolated vertex in } G(n,\tfrac{\lambda}{n}))\ 1(\text{vertex 2 is a isolated vertex in } G(n,\tfrac{\lambda}{n}))\right)\right]$$

$$+ n^2\left[\mathbb{P}\left(1(\text{vertex 1 is a isolated vertex in } G(n,\tfrac{\lambda}{n}))\ 1(\text{vertex 2 is a isolated vertex in } G(n,\tfrac{\lambda}{n}))\right) - \left(\mathbb{P}\left(1(\text{vertex 1 is a isolated vertex in } G(n,\tfrac{\lambda}{n}))\right)\right)^2\right]$$

Using $\mathbb{P}\left(1(\text{vertex 1 is a isolated vertex in } G(n,\tfrac{\lambda}{n}))\ 1(\text{vertex 2 is a isolated vertex in } G(n,\tfrac{\lambda}{n}))\right) = \left(1-\frac{\lambda}{n}\right)^{2n-2}$

(Ex)

$$\text{Var}(Y_n) \leq \mathbb{E}[Y_n] + \frac{\lambda}{n-\lambda}\left(\mathbb{E}[Y_n]\right)^2$$

□

Claim 2 : If $\lambda \geq a \log(n)$ for $\exists\ a > \frac{1}{2}$ then

$\mathbb{P}(\text{Erdös-Renyi}(n, \frac{\lambda}{n}), Y_n = 0) \longrightarrow 0$

as disconnected as $n \to \infty$

Assume claim

Theorem :- $\lambda - \log n \to \infty$ (Threshold)

$\mathbb{P}\left(\text{Erdös-Renyi}(n, \frac{\lambda}{n}) \text{ is Connected}\right) \to 1$

If $\lambda - \log n \to -\infty$

$\mathbb{P}\left(\text{Erdös-Renyi}(n, \frac{\lambda}{n}) \text{ is Connected}\right) \to 0$

Proof :- Suppose $\lambda - \log n \longrightarrow -\infty$

$\overset{\text{HW 12}}{\Rightarrow}$ - $E[Y_n] \longrightarrow \infty$ as $n \to \infty$. $-\circledast$

2nd moment method

$\mathbb{P}(Y_n = 0) \overset{\text{Ex}}{\leq} \mathbb{P}(|Y_n - E[Y_n]| \geq E[Y_n])$

$\leq \text{Var}(Y_n) / E[Y_n]^2$

$= \dfrac{E[Y] + \frac{\lambda}{n-\lambda}(E[Y])^2}{(E[Y])^2}$

$= \dfrac{2}{E[Y]} + \dfrac{\lambda}{n-\lambda}$

$\dfrac{2}{E[Y]} + \dfrac{\lambda}{n-\lambda} \longrightarrow 0$ as $n \to \infty$ by $\circledast$

by $\circledast$

$\therefore \lambda - \log n \to -\infty \Rightarrow \mathbb{P}(Y_n = 0) \to 0 - \oplus$ as $n \to \infty$.

Now

$$\mathbb{P}\left(\text{Erdös-Renyi}\left(n, \frac{\lambda}{n}\right) \text{ is connected}\right)$$

$$= \mathbb{P}(Y_n = 0) - \mathbb{P}(\text{Erdös-Renyi}\left(n, \frac{\lambda}{n}\right), Y_n = 0)$$

$$\leq \mathbb{P}(Y_n = 0)$$

$\therefore$ If $\lambda - \log n \to -\infty$ by (+)

$\Rightarrow \mathbb{P}\left(\text{Erdös-Renyi}\left(n, \frac{\lambda}{n}\right) \text{ is connected}\right) \to 0$ as $n \to \infty$

- If $\lambda - \log n \to \infty$, $\lambda \leq 2\log n$ — (*)

$$\mathbb{P}(Y=0) = 1 - \mathbb{P}(Y \geq 1) \geq 1 - E[Y]$$

(Markov)

$$\geq 1 - ne^{-\lambda} e^{\lambda/n}$$

And

$1 - ne^{-\lambda} e^{\lambda/n} \longrightarrow 1$ as $n \to \infty$ by (**)

$\therefore$

$$\mathbb{P}\left(\text{Erdös-Renyi}\left(n, \frac{\lambda}{n}\right) \text{ is connected}\right) \longrightarrow 1$$

as $n \to \infty$.

Ex: $\lambda \geq 2\log(n)$ then show that

$$\mathbb{P}\left(\text{Erdös-Renyi}\left(n, \frac{\lambda}{n}\right) \text{ is connected}\right) \longrightarrow 1$$ □

sketch of proof Claim 2

Take a realisation of Erdös-Renyi $(n, \frac{\lambda}{n})$

- $X_k \equiv$ # trees of size k which cannot be extended to $k+1$

  i.e $\{v_1, v_2, \dots, v_k\} \subseteq [1, \dots, n]$ contain a tree in Erdös-Renyi $(n, \frac{\lambda}{n})$ & $v \notin \{v_1, v_2, \dots, v_k\}$ there is no edge between $v \to v_i$ $i = 1, \dots, k$.

E.g  $n = 8$

$\Rightarrow$ $X_3 = 3$ $X_5 = 1$

Suppose Erdös-Renyi $(n, \frac{\lambda}{n})$ is disconnected but $Y_1 = 0$ $\exists\, k \in \{2, \dots, \frac{n}{2}\}$ for $X_k \geq 1$.

$$\mathbb{P}\left(\text{Erdös-Renyi } (n, \tfrac{\lambda}{n}) \text{ is disconnected}, Y_n = 0\right) \leq \mathbb{P}\left(\bigcup_{k=2}^{n/2} \{X_k \geq 1\}\right)$$

$$\leq \sum_{k=2}^{n/2} \mathbb{P}(X_k \geq 1)$$

$$\leq \sum_{k=2}^{n/2} \mathbb{E}[X_k]$$

Ex: $\mathbb{E}[X_k] = \binom{n}{k} k^{k-2} \cdot \left(\frac{\lambda}{n}\right)^{k-1} \left(1-\frac{\lambda}{n}\right)^{k(n-k)}$

- If $\exists\, a > \frac{1}{2}$ $\lambda \geq a \log n$

  then $\mathbb{E}[X_k] \leq n^{1-k/4}$ $k \geq 5$

$\therefore$ $\mathbb{P}($ Erdös-Renyi $(n, \frac{\lambda}{n})$, $Y_n = 0)$
n disconnected

$$\leq 4n(e\lambda)^4 e^{-\lambda k} + \sum_{k=5}^{n/2} n^{1-k/4}$$

$\longrightarrow 0$ as $n \to \infty$ (Ex) □