Definition: An event $A \subseteq \Omega$ is observable until time n when it can be written as a union of basic events of the form

$$\{\omega \in \Omega \mid \omega_1 = o_1, \ldots, \omega_n = o_n\} \quad o_i \in \{-1, 1\}, \ 1 \le i \le n$$

i.e. A can be determined from the outcome of the 1st n tosses.

$\mathcal{A}_n :=$ class of event A that can be observed by time n. [include ϕ]

$\{\phi, \Omega\} = \mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \ldots \subseteq \mathcal{A}_N \equiv \mathcal{F}$. The sequence $\{\mathcal{A}_n\}_{n=0}^{N}$ is called a filtration.

Notation: $A \subseteq \Omega$ (indicator of A)

$$1_A(\omega) = \begin{cases} 1 & \omega \in A \\ 0 & \omega \notin A \end{cases}$$

Definition: A map $T: \Omega \to \{0, 1, \ldots, N\} \cup \{\infty\}$ is called a stopping time if

$$\{T = n\} \equiv \{\omega \in \Omega \mid T(\omega) = n\} \in \mathcal{A}_n \quad 0 \le n \le N$$

$$\Leftrightarrow \{T \le n\} \in \mathcal{A}_n \quad 0 \le n \le N.$$

i.e. $\{T = n\}$ is an event observable until time n.

Example: For $a \in \mathbb{Z}$ let

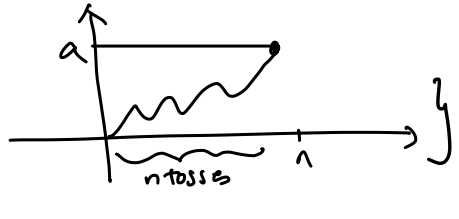
$$\sigma_a : \Omega \to \{0, 1, \ldots, N\} \cup \{\infty\}$$

$$\sigma_a(\omega) = \min \{k \in [0, N] \mid S_k(\omega) = a\}$$

$\min(\emptyset) = \infty$

Ex:- $\sigma_a(\cdot)$ is a stopping time

$\{\omega \in \Omega : \sigma_A(\omega) = n\}$

$=$ 

$= \{\omega \in \Omega \mid X_1(\omega) \neq a, \ldots, X_{n-1}(\omega) \neq a, X_n(\omega) = a\}$

$=$ .. "only depends on 1st n tosses"

Ex: $\min(\sigma_A, N)$ is a stopping time.

$n < N$ • $\{\omega \in \Omega : \min(\sigma_A, N) = n\}$

$= \{\omega \in \Omega : \sigma_A = n\} \in \mathcal{A}_n$

$n = N$ • $\{\omega \in \Omega : \min(\sigma_A, N) = n\} = \{\omega \in \Omega \mid \sigma_A \geq N\}$

$= \left(\bigcup_{n=1}^{N-1} \{\omega \in \Omega \mid \sigma_A = n\}\right)^c \in \mathcal{A}_N$

Theorem 1 :- For any stopping time $T: \Omega \to \{0, 1, \ldots, N\}$

[Impossibility of a favorite stopping]

$$E[S_T] = 0$$

$S_T(\omega) \equiv S_{T(\omega)}(\omega) \equiv$ outcome of the trajectory ω at stopping time $T(\omega)$

Proof :- T is a stopping time.

$k = 0, 1, 2, \ldots N$ $\quad \{T \geq k\} \in \mathcal{A}_{k-1}$ — (1)

$$\left[\because \{T \geq k\} = \left(\{T = 1\} \cup \ldots \{T = k-1\}\right)^c\right]$$

$\{T = 1\} \cup \ldots \{T = k-1\} \in \mathcal{A}_{k-1}$ & $\mathcal{A}_{k-1}$ is closed under compliments

$$S_T = \sum_{k=1}^{N} X_k \mathbb{1}(T \geq k) \quad \text{— (2)}$$

$$\left[\because \quad S_T = \sum_{k=1}^{N} S_k \mathbb{1}(T = k)\right.$$

$$= \sum_{k=1}^{N-1} S_k \left[\mathbb{1}(T \geq k) - \mathbb{1}(T \geq k+1)\right] + S_N \mathbb{1}(T = N)$$

$$= \sum_{k=1}^{N-1} S_k \mathbb{1}(T \geq k) - \sum_{k=1}^{N-1} S_k \mathbb{1}(T \geq k+1) + S_N \mathbb{1}_{T=N}$$

$$= S_1 \mathbb{1}_{T\geq 1} + \sum_{k=2}^{N-1} S_k \mathbb{1}_{(T\geq k)} + S_N \mathbb{1}_{T=N} - \sum_{k=1}^{N-1} S_k \mathbb{1}_{(T\geq k+1)}$$

$$= S_1 \mathbb{1}_{T\geq 1} + \sum_{m=1}^{N-2} S_{m+1} \mathbb{1}_{(T\geq m+1)} + S_N \mathbb{1}_{T=N} - \sum_{k=1}^{N-1} S_k \mathbb{1}_{(T\geq k+1)}$$

$$= S_1 \mathbb{1}_{T\geq 1} + \sum_{l=1}^{N-2} (S_{l+1} - S_l) \mathbb{1}_{T\geq l+1} + (S_N - S_{N-1}) \mathbb{1}_{T=N}$$

$\{T=N\} \equiv \{T\geq N\}$

$$= X_1 \mathbb{1}_{T\geq 1} + \sum_{l=1}^{N-2} X_l \mathbb{1}_{T\geq l+1} + X_N \mathbb{1}_{T=N} \Big]$$

$$\mathbb{E}[S_T] \overset{(2)}{=} \mathbb{E}\Big[\sum_{k=1}^{N} Y_k \mathbb{1}_{T\geq k}\Big]$$

linearity of $\mathbb{E}$ $\Leftarrow$

$$= \sum_{k=1}^{N} \mathbb{E}\big[Y_k \mathbb{1}_{T\geq k} \big]$$

Y_k: depend on the k^{th} toss $\rightarrow \omega_k$

$\mathbb{1}_{T\geq k} \in A_{k-1}$ $\rightarrow$ "$\omega_1, \ldots, \omega_{k-1}$"

independence $\overset{Ex}{\Leftarrow}$

$$= \sum_{k=1}^{N} \mathbb{E}[Y_k]\, \mathbb{E}[\mathbb{1}_{(T\geq k)}]$$

$$= \sum_{k=1}^{N} 0 \cdot P(T \geq k) = 0$$

□

Game System: $V_1, \ldots, V_N : \Omega_N \to \mathbb{R}$ are random variables such that

$$\{V_k = c\} \in \mathcal{A}_{k-1} \quad c \in \mathbb{R}, \quad k = 1, 2, \ldots N$$

[$V_k \equiv$ amount of money you will place as a bet in the k^{th} round

• Result in the k^{th} round $\equiv V_k X_k$]

$$S_n^V = \sum_{k=1}^{N} V_k X_k$$

Theorem 2: (impossibility of profitable games)
For any $V_1, V_2, \ldots, V_N$ game system then $E[S_N^V] = 0$

Proof: like before

$$E[S_N^V] = \sum_{k=1}^{N} E[V_k X_k] - (\#)$$

It is enough to show $E[V_k X_k] = 0$.

V_k: $\mathcal{A}_{k-1}$; X_k: depends on k^{th} toss

$\text{Range}(V_k) = \{c_1, c_2, \ldots, c_m\}$

$$\Rightarrow \quad X_k V_k = \sum_{i=1}^{m} c_i \, X_k \mathbb{1}(V_k = c_i)$$

$$E[X_k V_k] = \sum_{i=1}^{m} c_i \, E[X_k \mathbb{1}(V_k = c_i)]$$

Depends on: "ω_k" "$\omega_1, \ldots \omega_{k-1}$"

$$= \sum_{i=1}^{m} c_i \, E[X_k] \, \mathbb{P}(V_k = c_i)$$

$$= \ldots 0 \ldots = 0 \quad \hookrightarrow (\text{x})$$

Place (x) into (#) $\Rightarrow$ to get result □

Impact of Theorem 2:

$T: \Omega \to \{0, 1, \ldots, N\}$ is a stopping time

- $V_k = \mathbb{1}_{\{T \geq k\}}$

$$S'_N = \sum_{k=1}^{N} X_k V_k = \sum_{k=1}^{N} X_k \mathbb{1}_{(T \geq k)}$$

$$\overset{(2)}{=} S_T \qquad [\text{Theorem 2} \Rightarrow \text{Theorem 1}]$$

- $V_k = \underbrace{S_{k-1}}_{(\omega_1, \dots, \omega_{k-1})} \underbrace{1_{T \geq k}}_{①}$ (take this definition)

$\Rightarrow$ $V_1, V_2, \dots, V_N$ is a game system.

$$V_k X_k = S_{k-1} X_k 1_{T \geq k}$$

Observe: $S_k^2 = (S_{k-1} + X_k)^2$

$$= S_{k-1}^2 + 2 S_{k-1} X_k + X_k^2$$
$$= S_{k-1}^2 + 2 S_{k-1} X_k + 1$$

$$\Rightarrow V_k X_k = \frac{(S_k^2 - S_{k-1}^2 - 1)}{2} 1_{T \geq k}$$

$$S_N^V = \sum_{k=1}^{N} V_k X_k$$

$$= \sum_{k=1}^{N} \frac{(S_k^2 - S_{k-1}^2 - 1)}{2} 1_{T \geq k}$$

$$\overset{(X_k^2 = 1)}{=} \sum_{k=1}^{N} \frac{S_k^2 1_{T \geq k}}{2} - \sum_{k=1}^{N} \frac{S_{k-1}^2 1_{T \geq k}}{2} - \sum_{k=1}^{N} \frac{1_{T \geq k}}{2}$$

$$= \sum_{k=1}^{N} \frac{S_k^2 1_{T=k}}{2} - \frac{T}{2}$$

$$S_N^V = \frac{(S_T^2 - T)}{2} - (xxx)$$

Theorem 2 $\Rightarrow$ $E[S_N^V] = 0$

$\therefore$ By (xxx) for any $T: \Omega \to \{0,1,\dots N\}$

we have

$$0 = E[S_N^V] = E\left[\frac{S_T^2 - T}{2}\right]$$

$$\Rightarrow \quad E[S_T^2] = E[T] - (\text{hatched})$$

Theorem 1: $E[S_T] = 0$

(hatched) $\Rightarrow$ $Var[S_T] = E[T]$

1.3 Ruin Problem:

Two players. Each with capital $\begin{cases} a - \text{I} \\ b - \text{II} \end{cases}$

Interpret

$S_n \equiv$ gain of player 1

Ruin (of player 1)

$$\{\sigma_{-a} < \sigma_b ; \sigma_{-a} \le N\}$$

where, $\sigma_u = \min\{k : S_k = u\}$

Define: Ruin Probability

$$\gamma'_N = \mathbb{P}(\sigma_{-a} \leq \sigma_b, \ \sigma_{-a} \leq N)$$

$$T_N = \min(\sigma_{-a}, \sigma_b, N)$$

Ex: T_N is a stopping time

$\therefore$ Theorem 1 $\Rightarrow$ $E[S_{T_N}] = 0$ — (3)

$T_N = \sigma_{-a} \Rightarrow S_{T_N} = -a$

$T_N = \sigma_b \Rightarrow S_{T_N} = b$

$T_N = N \Rightarrow S_{T_N} = S_N$

— Use this in (3)

$$0 = -a\,\mathbb{P}(T_N = \sigma_{-a}) + b\,\mathbb{P}(T_N = \sigma_b) + E[S_N \mathbb{1}_{T_N = N})$$

As $T_N = \min(\sigma_{-a}, \sigma_b, N)$

$$0 = -a\,\mathbb{P}(\sigma_{-a} < \sigma_b \ \& \ \sigma_{-a} \leq N)$$

$$+ \; b \; \mathbb{P}(\sigma_b < \sigma_{-a} \leq \sigma_b \leq N)$$

$$+ \; \mathbb{E}\, S_N \left(1_{\min(\sigma_{-a}, \sigma_b) \geq N}\right)$$

$\Rightarrow$

$$0 = -a \; \mathbb{P}_N(\text{Ruin of Player 1})$$
$$+ b \; \mathbb{P}_N(\text{Ruin of Player 2})$$
$$+ \mathbb{E}\left[S_N \, 1_{\min(\sigma_{-a}, \sigma_b) \geq N}\right] \quad (4)$$

- $r^1 = \lim_{N \to \infty} \mathbb{P}_N(\text{Ruin of Player 1})$ } Existência
- $r^2 = \lim_{N \to \infty} \mathbb{P}_N(\text{Ruin of Player 2})$

(4) $\overset{\text{Ex}}{\Rightarrow}$ $\lim_{N \to \infty}$ $\quad 0 = -a\, r^1 + b\, r^2 + 0$

$$\Rightarrow \quad -a\, r^1 + b\, r^2 = 0 \quad - (5)$$

$$r^1 + r^2 = 1 \quad - (6)$$

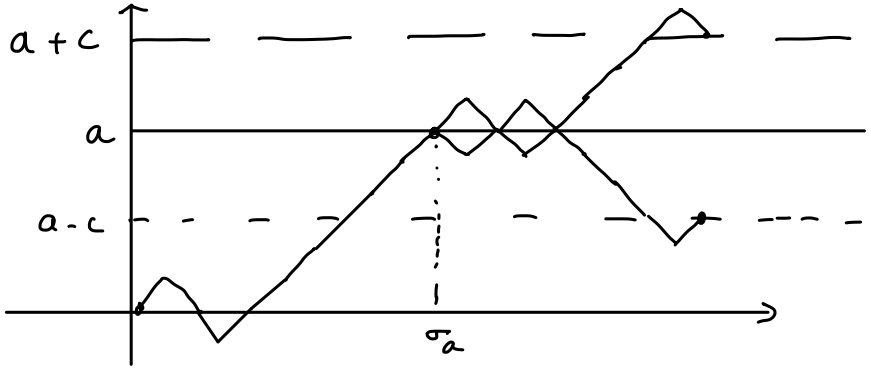
Ex

(5) e (6) $\quad r^1 = \frac{b}{a+b} \qquad r^2 = \frac{a}{a+b}$!

□

1.3 Reflection Principle

Picture Concept : $a \in \mathbb{N}$ $c \in \mathbb{N} \cup \{0\}$



Ex: For every path of the random walk after σ_a:

Each orange path that lands at $a+c$

there is a "Reflected" path across $y=a$

green path that lands at $a-c$

$$|\{\omega \in \Omega : S_n = a+c\}|$$

$$= |\{\omega \in \Omega : \sigma_a \le n,\ S_n = a+c\}|$$

$$= |\{\omega \in \Omega : \sigma_a \le n,\ S_n = a-c\}|$$

lemma 1 : $a, c \in \mathbb{N} \cup \{0\}$

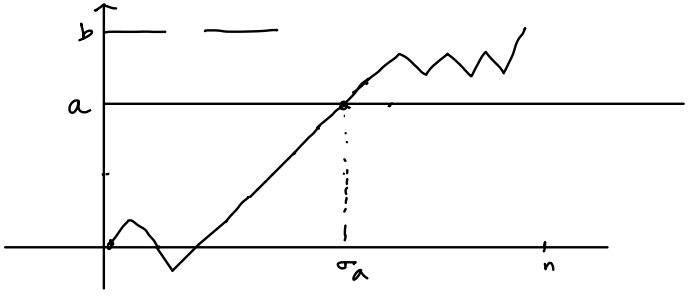
$$\mathbb{P}(S_n = a+c) = \mathbb{P}(\sigma_a \le n, S_n = a-c)$$

Theorem 2

(a) $\mathbb{P}(\sigma_a \le n) = \mathbb{P}(S_n \notin [-a, a-1])$

(b) $\mathbb{P}(\sigma_a = n) = \frac{1}{2}\left[\mathbb{P}(S_n = a-1) - \mathbb{P}(S_{n+1} = a+1)\right]$

Proof :- $\mathbb{P}(\sigma_a \le n) = \mathbb{P}\left(\sigma_a \le n, \bigcup_{b \in \mathbb{Z}} \{S_n = b\}\right)$



i.e. $\mathbb{P}(\sigma_a \le n) = \sum_{b \in \mathbb{Z}} \mathbb{P}(\sigma_a \le n, S_n = b)$

$$= \sum_{\substack{b \in \mathbb{Z} \\ b \ge a}} \mathbb{P}(\sigma_a \le n, S_n = b) + \sum_{\substack{b \in \mathbb{Z} \\ b < a}} \mathbb{P}(\sigma_a \le n, S_n = b)$$

Apply lemma 1.3.1
$b = a - c$
$(\Rightarrow a + c = 2a - b)$

Reflection Principle

$$= \sum_{\substack{b\in\mathbb{Z}\\ b\geq a}} \mathbb{P}(\sigma_a \leq n,\ S_n = b) + \sum_{\substack{b\in\mathbb{Z}\\ b<a}} \mathbb{P}(S_n = 2a-b)$$

(Ex)

$$= \sum_{\substack{b\in\mathbb{Z}\\ b\geq a}} \mathbb{P}(S_n = b) + \mathbb{P}(S_n > a)$$

$\{\sigma_a \leq n\} \subseteq \{S_n \geq b\}$

$$= \mathbb{P}(S_n \geq a) + \mathbb{P}(S_n > a)$$

$$\overset{\text{(Symmetry)}}{=} \mathbb{P}(S_n \geq a) + \mathbb{P}(S_n < -a)$$

$$= \mathbb{P}(S_n \notin [-a, a-1])$$

(b) $\mathbb{P}(\sigma_a = n) = \mathbb{P}(\sigma_a \leq n) - \mathbb{P}(\sigma_a \leq n-1)$

$\overset{\text{Ex}}{=}$ (Apply (a) + (Symmetry)) $\frac{1}{2}\left[\mathbb{P}(S_{n-1} = a-1) - \mathbb{P}(S_{n-1} = a+1)\right]$

□

Corollary 3 :- $a, n \in \mathbb{N}$

$$\mathbb{P}(\sigma_a = n) = \frac{a}{n}\,\mathbb{P}(S_n = a)$$

Proof: Earlier class :

$$\mathbb{P}(S_{n-1} = a-1) = \frac{1}{2^{n-1}} \binom{n-1}{\frac{(n-1)+(a-1)}{2}}$$

$$\mathbb{P}(S_{n-1} = a+1) = \frac{1}{2^{n-1}} \binom{n-1}{\frac{n-1+a+1}{2}}$$

$$\mathbb{P}(S_n = a) = \frac{1}{2^n} \binom{n}{\frac{n+a}{2}}$$

(Do the Combinatorics) $\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$

$$\binom{n-1}{k} = \frac{n-k}{k}\binom{n}{k}$$

Thm 2 (b) $\Rightarrow \mathbb{P}(\sigma_a = n) = \frac{a}{n}\mathbb{P}(S_n = a)$ □

$n \in \mathbb{N}$ $\quad \sigma_a = \min\{n \geq 1 \mid S_n = a\}$
Chitting time of a) $a \in \mathbb{Z}$ / hot

(Return time) $\sigma_0 = \min\{n \geq 1 \mid S_n = 0\}$

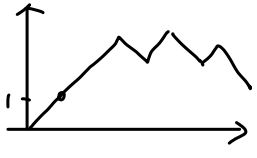
lemma 4: (Escape from Origin)

$$\mathbb{P}(\sigma_0 > 2n) = \mathbb{P}(S_{2n} = 0)$$

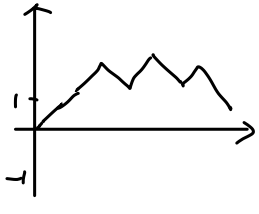
Proof:-

$$\mathbb{P}(\sigma_0 > 2n) = \mathbb{P}(S_1 \neq 0,\ S_2 \neq 0,\ \ldots,\ S_{2n} \neq 0)$$

$$= 2\,\mathbb{P}(S_1 > 0,\ S_2 > 0,\ \ldots,\ S_{2n} > 0) \quad \text{(Reflection principle)} \xleftarrow{\in r}$$



$$= 2 \cdot \frac{\#\text{ paths start at 1 \& stay positive for } 2n-1 \text{ steps}}{2^{2n}}$$



$$= \frac{1}{2^{2n-1}} \cdot \#\text{ paths start at 0 \& stay above } -1 \text{ for } 2n-1 \text{ step}$$

$$= \frac{1}{2^{2n-1}} \cdot \#\text{ paths start at 0 \& stay below 1 for } 2n-1 \text{ steps}$$

(Reflection Principle)

$$= 1 - \mathbb{P}(\sigma_1 \leq 2n-1)$$

Theorem 2 (a) with $a=1$ $\;\overset{\sim}{=}\; 1 - \mathbb{P}(S_{2n-1} \notin [-1, 0])$

$$\overset{\text{Ex}}{=} \mathbb{P}(S_{2n-1} = -1) = \mathbb{P}(S_{2n} = 0) \qquad \square$$

1.4 Arc Sine law - last visit to origin in $2N$ time

$$L = \max\{0 \leq n \leq 2N \mid S_n = 0\}$$

(not a stopping time)

"Order of L" : time after which one player has gain in capital

$$L \approx 2N$$

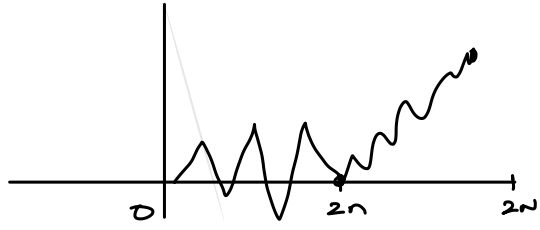
Theorem 5: $n \in \mathbb{N}$ not $n \leq N$

$$\mathbb{P}(L = 2n) = \frac{1}{2^{2N}} \binom{2n}{n} \binom{2N-2n}{N-n}$$

$$= \mathbb{P}(S_{2n} = 0)\, \mathbb{P}(S_{2N-2n} = 0)$$

Proof :-

# of paths of length $2N$ with $L = 2n$



$$= \left(\begin{array}{c}\#\text{ of paths of length } 2n \\ \text{with } S_{2n} = 0\end{array}\right) \times \left(\begin{array}{c}\#\text{ of paths of length } 2N-2n \\ \text{with } \sigma_0 > 2N-2n\end{array}\right)$$

1st $2n$ tosses — next $2N-2n$ tosses

$$\therefore \quad \mathbb{P}(L = 2n) = \mathbb{P}(S_{2n} = 0)\ \mathbb{P}(\sigma_0 > 2N-2n)$$

$$\left(\begin{array}{c}\text{lemma} \\ 4\end{array}\right) \leftarrow = \mathbb{P}(S_{2n} = 0)\ \mathbb{P}(S_{2N-2n} = 0)$$

$$= \binom{2n}{n} \frac{1}{2^{2n}} \binom{2N-2n}{N-n} \frac{1}{2^{2N-2n}}$$

$$= \underbrace{\frac{1}{2^{2N}} \binom{2n}{n} \binom{2N-2n}{N-n}}$$

Discrete arc Sine law

□

To understand order

$$\mathbb{P}\left(\frac{L}{2N} \le x\right) = \mathbb{P}(L \le 2Nx)$$

$$= \sum_{k=0}^{\lfloor Nx \rfloor} \mathbb{P}(L = 2k)$$

Ex 11.2 ← = stirling's approximation $n! \sim n^n e^{-n}$

$$\frac{1}{\pi} \frac{1}{\sqrt{n(N-n)}}$$

$$= \frac{1}{\pi} \cdot \frac{1}{2} \frac{1}{\sqrt{\frac{n}{2N}\left(1 - \frac{n}{2N}\right)}}$$

$$= \sum_{k=0}^{2x} \frac{1}{2} f\left(\frac{k}{2N}\right)$$

$f : [0,1) \to \mathbb{R}$

$f(x) = \frac{1}{\pi\sqrt{x(1-x)}}$

$N \to \infty$

$$\approx \int_0^x f(y)\, dy$$

$$= \int_0^x \frac{1}{\sqrt{y(1-y)}}\, dy$$

$$\mathbb{P}\left(\frac{L}{2N} \le x\right) \sim \frac{2}{\pi} \arcsin(\sqrt{x})$$