2 RECURRENCE AND TRANSIENCE

REMARK 2.1. (*Criteria for Recurrence and Transience*) As a consequence of the preceeding proposition, one way to decide whether a state $i \in S$ is recurrent or not is to check if $P_i(T_i < \infty) = 1$. That is, if a chain beginning in a state is guaranteed to return to that state (at some later point in time) the state is recurrent. If there is some chance the chain will not return to that state, the state is transient. Checking this condition may sometimes prove difficult so we shall next try to propose other equivalent criteria.

(a) $i \in S$ is recurrent if and only if

$$\sum_{n=0}^{\infty} p_{ii}^{[n]} = \infty. \tag{2.1}$$

Proof: See last week notes.

(b) If there are states $j, k \in S$ are such that $k \leftarrow j$ but $j \not\leftarrow k$, then j is transient.

Proof: See last week notes.

(c) If $i, j \in S$, $i \leftrightarrow j$ and i is recurrent, then j is also recurrent.

Proof: See last week notes.

Now that we have developed techniques on how to verify whether a state is recurrent or transient we shall focus on a decomposition result when the state space is finite.

THEOREM 2.2. *Let X_n be a Markov chain on S with transition matrix $\mathbf{P}$ and initial distribution μ. Suppose S is a finite set. Then S can be written as a disjoint union $T \cup (\cup_{k=1}^{m} R_k)$, where T is the set of all transient states and each $R_k, 1 \leq k \leq m$ is a closed communicating class of recurrent states.*

Proof : Let $T = \{j \in S|$ There is a $k \in S$ such that $k \leftarrow j$ and $j \not\leftarrow k\}$. By Remark 2.1 (b), all states in T are transient.

As $S \neq T$ (by Exercise **??**) there exists $z_1 \in S \backslash T$. Let $R_1 = \{x \in S : x \leftrightarrow z_1\}$. Note that $R_1 \cap T = \emptyset$. If $S = T \cup R$ the needed result has been shown. Otherwise select $z_2 \in S \backslash (T \cup R_1)$ let $R_2 = \{x \in S : x \leftrightarrow z_2\}$. We may continue to repeat this process and since S is a finite set, this process will end at a finite stage. So there exist $z_1, z_2, \ldots, z_m \in S \backslash T$ such that $R_k = \{x \in S : x \leftrightarrow z_k\}, k = 1, \ldots, m$ are disjoint closed communicating classes. and such that

$$S \backslash T = \cup_{k=1}^{m} R_k.$$

All that remains to be shown is that every state contained in one of the sets R_k is recurrent.

For any $y \in R_k$, let $n_0(y) \in \mathbb{N}$ be such that $p_{yz_k}^{[n_0(y)]} > 0$. Letting $c = \min_{y \in R_k} p_{yz_k}^{[n_0(y)]}$, we find that

$$\begin{aligned}
\sum_{y \in R_k} \sum_{m=1}^{\infty} p_{yy}^{[m]} &\geq \sum_{y \in R_k} \sum_{m=n_0(y)+1}^{\infty} p_{yy}^{[m]} \\
&\geq \sum_{y \in R_k} \sum_{m=n_0(y)+1}^{\infty} p_{yz_k}^{[n_0(y)]} p_{z_k y}^{[m-n_0(y)]} \qquad \text{(By (??))} \\
&= \sum_{y \in R_k} p_{yz_k}^{[n_0(y)]} \sum_{m=1}^{\infty} p_{z_k y}^{[m]} \\
&\geq c \sum_{y \in R_k} \sum_{m=1}^{\infty} p_{z_k y}^{[m]} \\
&= c \sum_{m=1}^{\infty} \sum_{y \in R_k} p_{z_k y}^{[m]} \\
&= c \sum_{m=1}^{\infty} 1 = \infty,
\end{aligned}$$

where, in the last step, we have used the fact that $p_{z_k x}^{[m]} = 0$ if $x \notin R_k$.

Since $\sum_{y \in R_k} \sum_{m=1}^{\infty} p_{yy}^{[m]}$ is infinite and since R_k is a finite set, that means there exists at least one $y \in R_k$ such that $\sum_{m=1}^{\infty} p_{yy}^{m} = \infty$. By Remark 2.1 (a), we conclude that there is at least one state in each R_k that is recurrent. So from Remark 2.1 (b) we conclude that all states in R_k are recurrent. ∎

3 HITTING PROBABILITIES AND EXPECTED HITTING TIMES

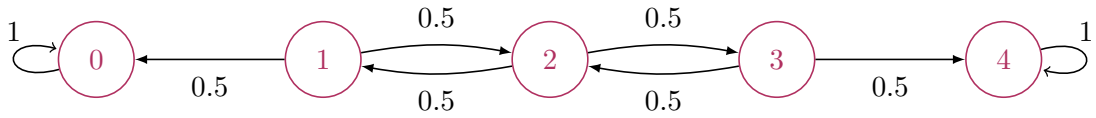
In this section we will analyze two problems which may be solved using similar techniques. The first problem is determining how likely it is that a Markov chain will visit one state before another (or whether it will ever visit a specified state at all). The second problem is determining how long it takes before a Markov chain reaches a certain state. We begin with some notation. Let X_n be a Markov chain on S with transition matrix $\mathbf{P}$ and initial distribution μ. If $i \in S$, we shall let P_i denote the probability P corresponding to the initial distribution $\mu(\{i\}) = 1$. Thus when distributed according to P_i the Markov chain starts in the state i, i.e $\mu(\{i\}) = 1 = P_i(X_0 = i)$.

Let us consider the following example.

EXAMPLE 3.1. Suppose Laxmi and Maria have two rupees each. They decide to play a game according to the following rules. At each turn a coin will be tossed, if it turns up heads then Laxmi will give Maria one rupee. If it turns up tails then Maria will give Laxmi one rupee. The game ends if any one player runs out of money. Let X_n be the wealth of, say, Laxmi at time n. It can be seen easily that X_n is a Markov chain on state space $S = \{0, 1, 2, 3, 4\}$ with $\mathbb{P}(X_0 = 2) = 1$ and transition matrix

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The above dynamics of the chain can be understood via a directed graph with S as the vertex set.



It is clear from above that $\{0\}$ and $\{4\}$ are closed communicating classes. Once the chain hits 0 (Maria wins the game) or the chain hits 4 (Laxmi wins the game) the chain remains there forever. Such states are called absorbing states. One natural question of interest is who wins the game. Another is how long the game will continue until there is a winner. Both questions may be reformulated in terms of the Markov chain. If the Markov chains visits 0 before visting 4 then Maria wins the game, but if 4 is visited first, then Laxmi wins. The value of n at the time the chain reaches either of these states is a random quantity, but it represents the game's duration. The answer to such questions requires analysis of the times the chain first visits certain sites. These are called hitting times and we will give precise definitions for this concept below. ■

DEFINITION 3.2. *Let X_n be a Markov chain on S with transition matrix $\mathbf{P}$ and initial distribution μ.*

(a) A state $j \in S$ is said to be absorbing if $\{j\}$ is a closed communicating class.

(b) Let $A \subset S$ and define the hitting time *of A as*

$$T^A = \min\{k \geq 0 : X_k \in A\}.$$

We will use the convention that $\min\{\emptyset\} = \infty$*, so either* T^A *is a non-negative integer equal to the first time the chain visits the set* A *or* $T^A = \infty$ *in which case no state in the set* A *is ever reached. Note that since* T^A *depends on the path of the Markov chain it is inherantly a random variable.*

EXAMPLE 3.3. (Example 3.1 Contd.) Let $A = \{4\}$. As we saw earlier 4 is an absorbing state. If the chain visits 4 then Laxmi wins the game. So the probability that Laxmi wins with initial capital 2 can be reformulated as

$$\mathbb{P}(X_n = 4 \text{ for some } n \geq 0 \mid X_0 = 2) = \mathbb{P}(T^A < \infty \mid X_0 = 2).$$

Finding the above probability involves a technique that uses a one-step decomposition of the chain to obtain a recursion relation. In this example we will first intuitively derive the recursion relation and also solve it.

Define a function $h : S \to [0, 1]$ to be given by $h(i) = \mathbb{P}(T^A < \infty \mid X_0 = i)$. This is the probability the chain ever reaches state 4 if it starts in state i, so we are interested in finding $h(2)$. First we observe

$$h(4) = \mathbb{P}(T^A < \infty, \mid X_0 = 4) \geq \mathbb{P}(T^A = 0 \mid X_0 = 4) = \mathbb{P}(X_0 = 4 \mid X_0 = 4) = 1.$$

As this quantity is a probability we also know $h(4) \leq 1$ which implies that

$$h(4) = 1. \tag{3.2}$$

Second we observe for all $n \geq 1$, $\mathbb{P}(X_n = 0 \mid X_0 = 0) = 1$ implying $P(X_n = 4 \text{ for some } n \geq 0 \mid X_0 = 0) = 0$. Therefore,

$$h(0) = 0. \tag{3.3}$$

Let us consider $h(2)$. Here the chain starts at 2 and wemust find the probability that it will reach 4. In one step the chain can either move to 1 or to 3 each with probability $\frac{1}{2}$. Once the chain reaches 1 or 3 then by Markov property it is enough to calculate the

probability of hitting 4 starting at 1 or 3. Thus by conditioning on the next step and using the Markov property we have

$$\begin{aligned} h(2) &= \mathbb{P}(X_n = 4 \text{ for some } n \geq 0 | X_0 = 2) \\ &= \mathbb{P}(X_n = 4 \text{ for some } n \geq 0 | X_0 = 2, X_1 = 1)\mathbb{P}(X_1 = 1 | X_0 = 2) \\ &\quad + \mathbb{P}(X_n = 4 \text{ for some } n \geq 0 | X_0 = 2, X_1 = 3)\mathbb{P}(X_1 = 3 | X_0 = 2) \\ &= \mathbb{P}(X_n = 4 \text{ for some } n \geq 0 | X_0 = 1)(\frac{1}{2}) + \mathbb{P}(X_n = 4 \text{ for some } n \geq 0 | X_0 = 3)(\frac{1}{2}) \\ &= \frac{1}{2}h(1) + \frac{1}{2}h(3). \end{aligned}$$

We do not know the values of $h(1)$ or $h(3)$ but doing the exactly similar conditioning on the next step and using the Markov Property we can see that

$$h(3) = \frac{1}{2}h(2) + \frac{1}{2}h(4) \tag{3.4}$$

$$h(1) = \frac{1}{2}h(2) + \frac{1}{2}h(0) \tag{3.5}$$

Using (3.2), (3.3), (3.4), and (3.5) in (3.4) we have

$$h(2) = \frac{1}{4}h(2) + \frac{1}{4}h(2) + \frac{1}{4}$$

which implies $h(2) = \frac{1}{2}$. So the probability that Laxmi will win the game is $\frac{1}{2}$. ■

The method of conditioning on one step and using the Markov property is widely applicable to caculate hitting distributions of sets in general. We present it in the next proposition.

PROPOSITION 3.4. *Let X_n be a Markov chain on S with transition matrix $\mathbf{P}$ and initial distribution μ. Let $A \subset S$ and T^A be the hitting time of the set A. Let $h^A : S \to [0,1]$ be given by*

$$h^A(i) = \mathbb{P}(T^A < \infty \mid X_0 = i).$$

(a) h^A is a solution of the linear system of equations given by

$$h^A(i) = \begin{cases} 1 & \text{if } i \in A \\ \sum_{j \in S} p_{ij} h^A(j) & \text{if } i \notin A \end{cases}$$

(b) If $f : S \to [0, \infty)$ is another solution to (a) then $f(i) \geq h(i)$ for all $i \in S$.

4 RANDOM WALK ON WEIGHTED GRAPHS

Let $G = (S, E)$ be a connected graphs. That is let S be a countable vertex set. Let

$$E \subset \{\{i, j\} | \, i, j \in V\}$$

be a set of edges for the graph. Let $\mu : E \to (0, \infty)$. As shorthand notations we define $\mu_{ij} = \mu(\{i, j\})$ for each edge $\{i, j\} \in E$ and $\mu_{ij} = 0$ if $\{i, j\}$ is not an edge. Note the edges are unordered pairs so that $\mu_{ij} = \mu_{ji}$. For each vertex $i \in V$ we further define

$$\mu_i = \sum_{\{i,j\} \in E} \mu_{ij}$$

Suppose that

$$0 < \mu_i < \infty.$$

for each $i \in V$, then we may define a matrix $\mathbf{P}$ with entries $p_{ij} = \frac{\mu_{ij}}{\mu_i}$ for all $i, j \in S$. Since each $p_{ij} \geq 0$ and since

$$\begin{aligned} \sum_{j \in S} p_{ij} &= \sum_{j \in S} \frac{\mu_{ij}}{\mu_i} \\ &= \frac{1}{\mu_i} \sum_{j \in S} \mu_{ij} \\ &= 1 \end{aligned}$$

then $\mathbf{P}$ is the transition matrix for a Markov chain on S. Therefore on any weighted graph G with a described initial distribtion X_0 on its vertices there is a canonical Markov chain X_n with state space S and transition matrix given by $\mathbf{P}$. Such a Markov chain is referred to as *Random walk on the weighted graph.*

PROPOSITION 4.1. *Suppose X_n is a random walk on a connected weighted graph G with weight function μ. If $\mu(S) < \infty$ then the walk is recurrent.*

Proof. Proof is not in syllabus

Let $j_0 \in S$ define $h : S \to [0, 1]$ be given by

$$h(i) = \mathbb{P}(T^{j_0} < \infty \mid X_0 = i),$$

where $T^{j_0} = \min\{n \geq 0 : X_n = j_0\}$. We have seen that h is a solution to

$$h(i) = \begin{cases} 1 & \text{if } i = j_0 \\ \\ \sum_{j \in S} p_{ij} h(j) & \text{if } i \neq j_0 \end{cases}$$

Let $\phi = 1 - h$. Then ϕ is a solution to

$$\phi(i) = \begin{cases} 0 & \text{if } i = j_0 \\ \\ \sum_{j \in S} p_{ij} \phi(j) & \text{if } i \neq j_0 \end{cases} \tag{4.6}$$

We claim that the energy or the Dirichlet form of ϕ is 0, i.e.

$$\mathcal{E}(\phi, \phi) := \frac{1}{2} \sum_{i \in S} \sum_{j \in S} \mu_{ij} (\phi(j) - \phi(i))^2 = 0. \tag{4.7}$$

If we assume the above then this readily implies that ϕ is constant but $\phi(j_0) = 0$. Therefore $\phi(i) = 0$ for all $i \in S$ which implies that $h(i) = 1$ for all $i \in S$ and that the walk is recurrent. We now prove (**??**). Observe that as $0 \leq \phi \leq 1$

$$\sum_{i \in S} \sum_{j \in S} \mu_{ij} \mid \phi(i) \mid \mid \phi(i) - \phi(j) \mid \leq \mu(S) < \infty$$

Consequently if S is countable (immediate if S is finite) using that the above series converges absolutely, we may rearrange the terms in the series to obtain

$$\begin{aligned} \frac{1}{2} \sum_{i \in S} \sum_{j \in S} \mu_{ij} (\phi(j) - \phi(i))^2 &= \frac{1}{2} \sum_{i \in S} \sum_{j \in S} \mu_{ij} \phi(j)(\phi(j) - \phi(i)) - \frac{1}{2} \sum_{i \in S} \sum_{j \in S} \mu_{ij} \phi(i)(\phi(j) - \phi(i)) \\ &= - \sum_{i \in S} \sum_{j \in S} \mu_{ij} \phi(i)(\phi(j) - \phi(i)) \\ &= - \sum_{j \in S} \mu_{j_0 j} \phi(j_0)(\phi(j) - \phi(j_0)) - \sum_{i \in S, i \neq j_0} \sum_{j \in S} \mu_{ij} \phi(i)(\phi(j) - \phi(i)) \\ &= -\phi(j_0) \sum_{j \in S} \mu_{j_0 j} (\phi(j) - \phi(j_0)) - \sum_{i \in S, i \neq j_0} \phi(i) \mu_i \left(\sum_{j \in S} p_{ij} \phi(j) - \phi(i) \right) \end{aligned}$$

Using (**??**) we have that each of the terms above ie equal to 0.

■