**Due: Wednesday, September 2nd, 2026, 8am**

1. Let $(X_i)$ be i.i.d. random variables with $\mathbb{P}(X_1 = 0) = \mathbb{P}(X_1 = 1) = \frac{1}{2}$. Let $S_n = \sum_{i=1}^n X_i$.

   (a) Let $a \in (\frac{1}{2}, 1]$. Show that

   $$2^{-n} \max_{k \geq an} \binom{n}{k} \leq \mathbb{P}(S_n \geq an) \leq (n+1)2^{-n} \max_{k \geq an} \binom{n}{k}$$

   (b) Using Stirling's formula find the $\lim_{n\to\infty} \max_{k \geq an} \binom{n}{k}$.

   (c) Show that for all $a > \frac{1}{2}$

   $$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(S_n \geq an) = -I(a),$$

   where

   $$I(z) = \begin{cases} \log 2 + z \log z + (1-z)\log(1-z) & z \in [0,1] \\ \infty & \text{otherwise.} \end{cases} \tag{1}$$

   (d) Using symmetry, show that for $a < \frac{1}{2}$

   $$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(S_n \leq an) = -I(a),$$

   where $I$ is as in (1)

   (e) Conclude that for any $\delta > 0$, we have

   $$\mathbb{P}(|\ \frac{S_n}{n} - \frac{1}{2}\ | \geq \delta) < \infty$$

   and consequently $\mathbb{P}(\frac{S_n}{n} \to \frac{1}{2} \text{ as } n \to \infty) = 1$.

2. Let $X_n$ be a Markov chain on $S$ with transition matrix $P$ and initial distribution $\mu$. A state $j \in S$ is said to be accessible from another state $i \in S$ if there is an $n \geq 0$ such that $p_{ij}^{[n]} > 0$. This is denoted by $j \leftarrow i$.

   $X_n$ is said to be irreducible if $i \leftrightarrow j$ for all $i, j \in S$

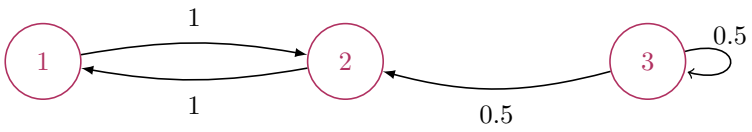
   and

   the period of $i \in S$ is defined as $d(i) = g.c.d.\{m \geq 1 : p_{ii}^{[m]} > 0\}$.

   (a) $\leftrightarrow$ defined above is an equivalence relation on $S$.

   (b) For $i, j \in S$, if $i \leftrightarrow j$ then show that $d(i) = d(j)$.

   (c) Consider a Markov chain $X_n$ on state space $S = \{1, 2, 3\}$, and whose transitions are given via the directed graph

   i. Find the transition matrix $\mathbf{P}$ for the chain.
   ii. Find the period $d(i)$ for each $i \in S$
   iii. Show that the chain is irreducible.