Wednesday, September 23rd, 2026
Time: 2:00-5:00pm

Today's will be a worksheet class. We will learn the following topics via examples and computations:

1. Reversible chains and simulating a uniform random variable on an arbitrary (large) finite set S using Markov Chains.
2. You will need the following definition and results:

   Let X_n be a Markov Chain on S with transition matrix $\mathbf{P}$ and initial distribution μ.

   (a) A state $j \in S$ is said to be accessible from another state $i \in S$ if there is an $n \geq 0$ such that $p_{ij}^{[n]} > 0$. This is denoted by $j \leftarrow i$.

   (b) Two states i and j in S are said to communicate if i and j are each accessible from the other. This is denoted by $j \leftrightarrow i$. X_n is said to be irreducible if $i \leftrightarrow j$ for all $i, j \in S$.

   (c) The period of $i \in S$ is defined as

   $$d(i) = g.c.d.\{m \geq 1 : p_{ii}^{[m]} > 0\}$$

   If $p_{ii}^{[m]} = 0$ for all $m \geq 1$, then $d(i)$ is defined to be 0. Finally, X_n is said to be aperiodic if $d(i) = 1$ for all $i \in S$.

   (d) A probability measure π on S is said to be a stationary distribution for the Markov chain X_n if for all $j \in S$,

   $$\sum_{i \in S} \pi(i) p_{ij} = \pi(j).$$

   Further, π is said to be *reversible* if

   $$\pi(\{i\}) p_{ij} = \pi(\{j\}) p_{ji} \quad \text{for every } i, j \in S.$$

Instructions:

- This will be no internet, no device (phone, PC tablet, laptop etc) class. You may use notes and any hand written material.
- Each worksheet sesssion:
  - 1st 25 minutes work individually (no consulting any one),
  - second 15 minutes you may consult with anyone, and
  - final 10 minutes work individually. Each period will be announced.
- Write your own solutions and do not copy solutions from anyone else. Answer each question as best as you can.
- The weightage for the scores on the worksheet will be equivalent to one quiz.

Wednesday, September 23rd, 2026
Time: 2:10-3:00pm

1. (*Random walk on the weighted graph* (G, μ).) Let $G = (S, E)$ be a connected graph. Assume $0 <\mid S \mid< \infty$ and $E \subset \{\{i, j\} \mid i, j \in S\}$ be a set of edges for the graph. Let $\mu : E \to (0, \infty)$ be the weight function. As shorthand notations we define $\mu_{ij} = \mu(\{i, j\}) > 0$ for each edge $\{i, j\} \in E$ and $\mu_{ij} = \mu(\{i, j\}) = 0$ if $\{i, j\}$ is not an edge. Note that the edges are unordered pairs so that $\mu_{ij} = \mu_{ji}$. For each vertex $i \in S$ we further define

$$\mu_i = \sum_{\{i,j\} \in E} \mu_{ij}$$

Define the transition matrix $\mathbf{P}$, for the Markov chain $\{X_n\}_{n \geq 0}$ on S with entries

$$p_{ij} = \frac{\mu_{ij}}{\mu_i}$$

for all $i, j \in S$. Find the *reversible* probability measure π for $\{X_n\}_{n \geq 0}$.

2. Let $\{X_n\}_{n \geq 0}$ be a Markov chain on S with transition matrix $\mathbf{P}$ and initial distribution η. Let π be a reversible probability measure for $\{X_n\}_{n \geq 0}$. Show that for any $n \geq 1$ and $i, i_1, i_2, \ldots i_{n-1}, j \in S$ we have

$$\begin{aligned} &\pi(\{i\})\mathbb{P}(X_1 = i_1, X_2 = i_2, \ldots, X_{n-2} = i_{n-2}, X_{n-1} = i_{n-1}, X_n = j \mid X_0 = i) \\ &= \pi(\{j\})\mathbb{P}(X_1 = i_{n-1}, X_2 = i_{n-2}, \ldots, X_{n-2} = i_2, X_{n-1} = i_1, X_n = i \mid X_0 = j) \end{aligned}$$

Draw a schematic picture for $n = 3$ case and explain the term reversible in words.

3. Let $S = \{0, 1\}^m$ for $m \geq 1$. Let X_0 be any point in S. Suppose we are given X_k for $0 \leq k \leq n$ for some $n \geq 0$ and let $X_k = (s_1, s_2, s_3, \ldots, s_m) \in S$. Choose U to be a uniformly chosen number in $\{1, 2, 3, \ldots, m\}$. Define

$$X_{k+1} = (s_1, s_2, \ldots, 1 - s_U, \ldots s_m),$$

i.e. flip the U-th coordinate of X_k.

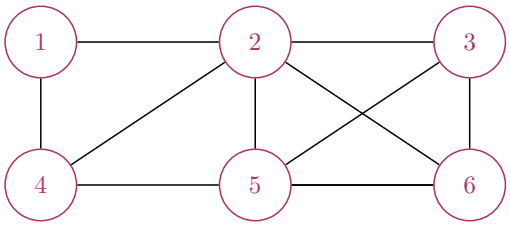
   (a) Find $\mathbf{P}$ such that X_n is a Markov chain on S with a transition matrix $\mathbf{P}$.

   (b) Find the stationary distribution π for $\{X_n\}_{n \geq 0}$.

   (c) How will you use this chain to simulate uniform random variable on S ?

Due: Wednesday, September 23rd, 2026
Time: 3:40-4:30pm

1. Consider the weighted graph $G = (S, E)$ with vertex set $S = \{1, 2, 3, 4, 5, 6\}$ and edge set as given in the figure below.



   (a) Let μ denote the natural weights i.e $\mu \equiv 1$ on all edges. Let $\{X_n\}_{n\geq 0}$ be the random walk on the weighted graph (G, μ). Find the stationary distribution for $\{X_n\}_{n\geq 0}$.

   (b) Let $\{X_n\}_{n\geq 0}$ be a Markov chain on S with transition matrix $\mathbf{P}$ and initial distribution η. Suppose $\mathbf{P}$, on the vertex set S given by

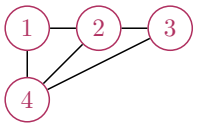
$$p_{ij} = \begin{cases} \frac{1}{5} & \text{if } \{i,j\} \text{ is an edge in } G \\ 0 & \text{if } \{i,j\} \text{ is not an edge in } G, i \neq j \\ 1 - \frac{\text{degree of } i}{5} & \text{if } i = j \end{cases}$$

   Find the stationary distribution of $\{X_n\}_{n\geq 0}$

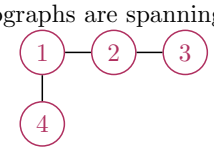
   (c) What is the significance of 5? What are the differences between the Markov chains in (a) and (b) ?

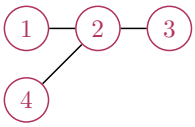
Due: Wednesday, September 23rd, 2026
Time: 5:10-8:00pm
Submit on Moodle, if you use internet tools then please acknowledge

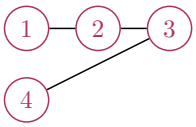
A spanning tree of a connected graph $G = (V, E)$ is a connected subgraph of G whose vertex set is V and has no cycles. A cycle is a (finite) connected subgraph C such that every vertex of C is the endpoint of exactly two edges of C.

For example: if G is the graph: 

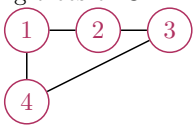
The following subgraphs are spanning trees of G:

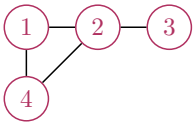






The following subgraphs are not spanning trees of G :





as they contain a cycle.

1. Let $G = (V, E)$ be a connected graph. Assume $0 <\mid V \mid< \infty$ and $E \subset \{\{k, l\} \mid k, l \in V\}$ be a set of edges for the graph. Let S be the set of all spanning trees of G Assume that we are given and $X_0 \in S$ and X_k for $0 \leq k \leq n$ for some $n \geq 0$. Construct X_{k+1} via the following:

   (i) Randomly (and uniformly) choose an edge $e \in E$ that is not in X_k.

   (ii) Let C denote the unique cycle of $X_k \cup \{e\}$.

   (iii) Choose an edge f uniformly at random from the edges in C.

   (iv) Set $X_{k+1} = X_k \cup \{e\} \setminus \{f\}$.

   You may assume the above defines a Markov chain on S with an appropriate transition matrix $\mathbf{P}$. Using the facts[1] about spanning trees:

   (a) Suppose $i, j \in S$. Assuming that $j = (i \cup \{e\}) \setminus \{f\}$ for some $e, f \in E$ and $e \neq f$, find an expression for p_{ij} and p_{ji}.

   (b) Show that $\{X_k\}_{k \geq 0}$ is an aperiodic, irreducible, Markov chain on S.

   (c) Find the stationary distribution π of $\{X_n\}_{n \geq 0}$.

   (d) Can you describe the importance of the above chain ?

[1] Every spanning tree of Ghas exactly $\mid V \mid -1$ edges. Every subgraph of G with $\mid V \mid -1$ edges and no cycles is a spanning tree. Adding an edge to a spanning tree always creates exactly one cycle.