Ordinary Differential Equations (ODEs) – part 1

Complex Analysis – part 2

- A differential equation, whose unknown function is a function of a single variable.
- All derivatives are w.r.t that single variable.
- ORDER: - determined by the order of the highest derivative present in the differential equation.

Derivative: Rate of change of one quantity w.r.t. another.

Examples:
- chemical Reactions
- radioactive decay involving Spontaneous decomposition (rate of decay α amount of substance)

let $x: [0,T] \to \mathbb{R}$

$x(t) \equiv$ denote the amount of substance at time t

1st order ODE

$$\frac{dx}{dt} = -kx(t) \qquad k>0, \; t \in [0,T]$$

$$x(0) = x_0$$

Assume $x(0) > 0$ and $x(t) > 0 \quad \forall t \in [0,T]$

$\Rightarrow$ $x(\cdot)$ solves

$$\frac{1}{x(t)} \frac{dx}{dt} = -k \qquad \forall t \in [0,T]$$

$$\Rightarrow \quad \frac{d}{dt} \left(\log (x(t)) \right) = -k$$

Ex

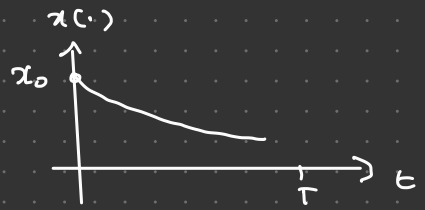
$$\Rightarrow \quad \log x(t) = -kt + C$$

$$\Rightarrow \quad x(t) = e^{C} e^{-kt}$$

At $t=0$ $\quad x(0) = x_0$

$$\Rightarrow \quad x(t) = x_0 e^{-kt}$$

$t \in [0,T]$



Example 2: A bank offers $r\%$ interest compounded n times in a year

$x_n(t) \equiv$ amount of money accumulated in t-years

$x_0 \equiv$ initial amount of money placed in the bank.

$$\Rightarrow \quad x_n(t) = \left(1 + \frac{r}{n}\right)^{nt} x_0$$

Suppose it is compounded continuously ("$n \to \infty$")

$$x_n(t) \longrightarrow e^{rt} x_0 \qquad t \geq 0$$

$$\| \quad x(t)$$

$$\frac{dx}{dt} = r\, x(t) \qquad t \geq 0 \qquad \left(\begin{matrix}\text{growth} \\ \text{model}\end{matrix}\right)$$

$$x(0) = x_0$$

General nth order ODE

replace: $D \subseteq \mathbb{R}$

$$\phi : \mathbb{R}_+ \times \mathbb{R}^{n+1} \longrightarrow \mathbb{R} \qquad x : [0, \infty) \longrightarrow \mathbb{R}$$

$$\phi\left(t, x(t), x^{(1)}(t), x^{(2)}(t), \ldots, x^{(n)}(t)\right) = 0$$

where $x^{(k)}(t) = \dfrac{d^k}{dt^k} x(t)$

Known: $\phi(\cdot)$ unknown: $x(\cdot)$

Example: $\phi(t, x, x^{(1)}) = 0$ (1^{st} order ODE)

Typically

$$\frac{d^2x}{dt^2} = g\left(t, x(t), x^{(1)}(t)\right)$$

$g : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}$ (known)

2^{nd} order ODE in Normal form

Example 3 :- let a particle of mass m fall freely under gravity.

(a) $x : [0, \infty) \to \mathbb{R}$

$x(t)$ - distance traversed by the particle in time t

g - acceleration due to gravity

$$m \frac{d^2x}{dt^2} = mg \qquad t > 0$$

(Ex) $$\left.\frac{dx}{dt}\right|_{t=0} = v_0 \quad , \; x(0) = x_0$$

$$\Rightarrow \quad x(t) = \frac{1}{2} g t^2 + v_0 t + x_0.$$

(b) Air drag for the particle $\equiv -bv^2$

(v - velocity)

$v : [0, \infty) \to \mathbb{R}$ velocity of particle

$$m \frac{dv}{dt} = mg - bv^2 \qquad - (*)$$

There is a terminal velocity v_0 as $t \to \infty$

i.e. $$\frac{dv}{dt} = 0$$

Formally: $\circledast \Rightarrow mg - bv^2 = 0$

$$\Rightarrow v_0 = \sqrt{\frac{mg}{b}}$$

Re write: $\circledast$

$$\frac{m}{b}\frac{dv}{dt} = \frac{mg}{b} - v^2$$

$$\Rightarrow \frac{m}{b}\frac{dv}{dt} = v_0^2 - v^2$$

XII - way

$$\frac{dv}{v_0^2 - v^2} = \frac{m}{b}\,dt$$

Integrate both sides (from 0 to t)

(Ex) $$\frac{1}{2v_0}\ln\left(\frac{v_0+v}{v_0-v}\right) = \frac{b}{m}t$$

$\Rightarrow$ (Ex.)

$$v(t) = v_0 \tanh\left(\frac{t}{\sqrt{\frac{m}{gb}}}\right)$$

First order Differential Equation

$x: [0,a) \to \mathbb{R}$ $\quad f: [0,\infty) \times \mathbb{R} \to \mathbb{R}$ given continuous function

1st order ODE

$$\frac{dx}{dt}(t) = f(t, x(t)) \quad \forall\, t \geq 0$$

$$x(0) = x_0$$

a procedure for prescribing $x(\cdot)$

- the procedure makes sense if it delivers a unique $x(\cdot)$

$$x(t) = x_0 + \int_0^t f(s, x(s))\, ds$$

Separation of variables :- (1st order)

$x : [0, \infty) \to \mathbb{R}$ satisfies the ODE

$f : [0, \infty) \times \mathbb{R} \to \mathbb{R}$

$$f(t, x) = \frac{h(t)}{g(x)} \quad \text{for}$$

Some $h : [0, \infty) \to \mathbb{R}$

$g : \mathbb{R} \to \mathbb{R} \setminus \{0\}$

Example (velocity).

$$g(v) = \frac{1}{v_0^2 - v^2} \; ; \quad h(t) = \frac{m}{b}$$

- $\frac{dx}{dt} = \frac{h(t)}{g(x)}$ ... $g(x)\,dx = h(t)\,dt$

$$\int_{x_0}^{x} g(s)\,ds = \int_{t_0}^{t} h(s)\,ds$$

- $x(\cdot) = \cdot\cdot$ ... ✓ ?

let $h : [0,\infty) \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ let

$x : [0,\infty) \to \mathbb{R}$ satisfy Separable ODE

$$-h(t) + g(x(t))\,\frac{dx}{dt}(t) = 0 \qquad \forall\ t > 0$$

$$x(0) = x_0$$

Step 1:

$$h(t) = g(x)\,\frac{dx}{dt}(t) \qquad \forall\ t \geq 0$$

Integrate both sides w.r.t t

$$\textcircled{+} \quad \int_0^T h(t)\,dt = \int_0^T g(x(t))\,\frac{dx}{dt}(t)\,dt$$

let $G : [0,\infty) \to \mathbb{R}$ be such that $G'(x) = g(x)$

Assume antiderivative exists for $g(\cdot)$

Chain rule:

$$\frac{d}{dt} h(x(t)) = G'(x(t)) \frac{dx(t)}{dt} = g(x(t)) \frac{dx(t)}{dt}$$

$$\boxed{\frac{d}{dt} h(x) = g(x) \frac{dx}{dt}} \quad - (\dagger)$$

Substitute $(\dagger)$ in $(+)$ to get

$$\int_0^T h(t)\, dt = \int_0^T \left[\frac{d}{dt} h(x(t))\right] dt$$

$$= h(x(T)) - h(x(0))$$

$$= \int_{x(0)}^{x(T)} G'(y)\, dy$$

$$\Rightarrow \boxed{\int_0^T h(t)\, dt = \int_{x(0)}^{x(T)} g(y)\, dy}$$