Recall: Basic idea of ODE's

$x : [0,\infty) \longrightarrow \mathbb{R}^d$ $\quad d \geq 1$

(IVP)

$$\frac{dx}{dt} = f(t, x(t)) \qquad t > 0$$

$$x(0) = \alpha$$

$f : \mathbb{R}_+ \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$

$$x(t) = \alpha + \int_0^t f(s, x(s))\, ds$$

OMIT Proof — hypothesis on f to guarantee a solution.

Separable Variable

Case :-

$$f(x,t) = \frac{h(t)}{g(x)}$$

$g : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ $\quad d=1$

$h : [0,\infty) \to \mathbb{R}$

Solution to (IVP)

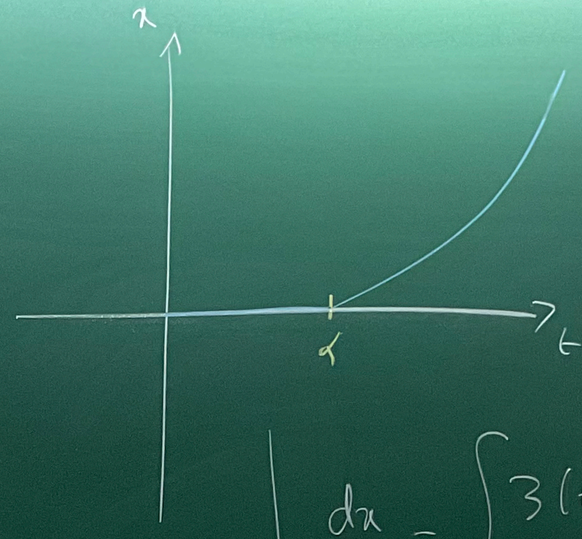
$$\int_{x(0)}^{x(t)} g(x)\, dx = \int_0^t h(s)\, ds$$

$$\int g(x)\, dx = \int h(t)\, dt$$

Example 2 :- $f: \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$

$$f(s,x) = 3x^{2/3}$$

$$\boxed{\begin{aligned} \frac{dx}{dt} &= 3x^{2/3} \\ x(0) &= 0 \end{aligned} \quad (*)}$$



infinitely many solutions !

$|h(x)-h(y)| \le c|x-y|$ (Lipschitz)

Let $\alpha > 0$

$$x_\alpha(t) = \begin{cases} (t-\alpha)^3 & t > \alpha \\ 0 & t \le \alpha \end{cases}$$

$$\frac{dx}{dt} = \begin{cases} 3(t-\alpha)^2 & t > \alpha \\ 0 & t \le \alpha \end{cases}$$

(Ex)

$$\frac{dx}{dt} = \begin{cases} 3x^{2/3} & t > \alpha \\ 3x^{2/3} & t \le \alpha \end{cases}$$

x solves $(*)$

Proof

$c|x-y|$

$.)$

Theorem 1: $f: [0,\infty) \times \mathbb{R} \to \mathbb{R}$

such that $\exists\, c > 0$

$$|f(t,x) - f(t,y)| \leq c|x-y|$$

$\forall\ t \in [0,T]$ (for any $T > 0$)

then $\dfrac{dx}{dt} = f(t, x(t))\ ,\ t > 0$

$x(0) = \alpha$

$x_n \to x$ in X-norm

has a unique solution.

Proof:- $x_0(t) = \alpha\ \forall t \geq 0$; $x_n(t) = \alpha + \int_0^t f(s, x_{n-1}(s))\,ds$

Theorem 2. (Gronwall's lemma)

Let $\eta : [0, \infty) \to \mathbb{R}$ be a continuous function satisfying

$$0 \leq \eta(t) \leq \alpha(t) + \beta \int_0^t \eta(s)\, ds \qquad -(*)$$

Then $\eta(t) \leq \alpha(t)\, e^{\beta t}$

Quantualisation $\alpha \equiv \alpha(t)$

Consequences: $\alpha = 0$, $\eta \equiv 0$

Ex. Show uniqueness in Theorem 1.

Pf.

$h(t) =$

$\Rightarrow \frac{dh}{dt} =$

$(*) \Rightarrow$

$\frac{dh}{dt} \leq$

$\Rightarrow \frac{dh}{d}$

Proof. $h : [0,\infty) \to \mathbb{R}$

$$h(t) = e^{\beta t} \int_0^t g(s)\,ds$$

$$\Rightarrow \frac{dh}{dt} = -\beta h(t) + e^{\beta t} g(t)$$

(*) $\Rightarrow$

$$\frac{dh}{dt} \le -\beta h(t) + e^{-\beta t}\left(\alpha + \beta \int_0^t g(s)\,ds\right)$$

$$\Rightarrow \frac{dh}{dt} \le -\beta h(t) + \alpha e^{-\beta t} + \beta h(t)$$

$$\Rightarrow \frac{dh}{dt} \le \alpha e^{-\beta t}$$

Integrate

$$h(t) - h(0) \le \int_0^t \alpha e^{-\beta s}\,ds$$

$$= \alpha \left. \frac{e^{-\beta s}}{-\beta} \right|_0^t$$

$$= \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right)$$

$\Rightarrow$

$$e^{-\beta t}\int_0^t q(s)\,ds - 0 \le \frac{\alpha}{\beta}\left(1-e^{-\beta t}\right)$$

$\Rightarrow$ $\boxed{\int_0^t q(s)\,ds \le \frac{\alpha}{\beta}\left(e^{\beta t}-1\right)}$ (**)

Place (**) into (*)

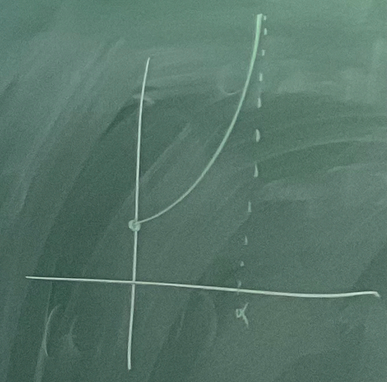
$$0 \le q(t) \le \alpha + \alpha\left(e^{\beta t}-1\right)$$

$\Rightarrow$ $0 \le q(t) \le \alpha e^{\beta t}$ $\square$

Example 3

$$x(t) = \frac{1}{\alpha - t} \qquad 0 \le t < \alpha$$

$\alpha > 0$



$0 \le t < \alpha$

$$\frac{dx}{dt} = \frac{-1}{(\alpha-t)^2}(-1)$$

$$= \frac{1}{(\alpha-t)^2} = (x(t))^2$$

$$\boxed{\frac{dx}{dt} = x^2 \quad t>0 \qquad x(0) = 1/\alpha}$$

$\Longleftrightarrow$ finite time blow up

$$x(t) = \frac{1}{t-\alpha} \qquad 0 \le t$$

Bernoulli Equations $m \in \mathbb{R}$ $x : [0, \infty) \to \mathbb{R}$

$$(*) \begin{cases} \dfrac{dx}{dt} + p(t)\, x(t) + q(t)\, x(t)^m = 0 \\ x(0) = \alpha \end{cases}$$

$p, q : [0, \infty) \to \mathbb{R}$

$m =$

$\dfrac{dx}{dt}$

Suppos

$n=0$ & $q \equiv 0$

$$\frac{dx}{dt} + px = \left[\frac{d}{dt}\left(e^{\int_0^t p(s)} x(t)\right)\right] e^{-\int_0^t p(s)ds}$$

Suppose x solves (*) then

$$\frac{d}{dt}\left(e^{\int_0^t p(s)} x(t)\right) = 0$$

$$\Rightarrow e^{\int_0^t p(s)} x(t) = c_1$$

$$x(t) = c_1 e^{-\int_0^t p(s)ds}$$

$$x(0) = \alpha$$

$$\Rightarrow \boxed{x(t) = \alpha e^{-\int_0^t p(s)ds}}$$

- $M=0$

Ex:

$$x(t) = e^{-\int_0^t p(s)ds}\left[\alpha + \int_0^t q(s) e^{\int_0^s p(r)dr} ds\right]$$

- $M=1$ ✓

General n: $z(t) = x(t)^\alpha$; find α to solve for z

Example 4 .. (logistic growth)

$N(t)$ = population at time t

$$\frac{dN}{dt} = f(N)N$$

$f(N) \simeq r$ near 0 & $f(N) << 0$ $N \to \infty$

Eg: $f(N) = r - aN$ $\quad r>0, a>0$

$$\frac{dN}{dt} = rN - aN^2 \quad , N(0) = P$$

$$\Rightarrow \boxed{N(t) = \frac{\frac{r}{a} P}{P + \left(\frac{r}{a} - P\right) e^{-rt}}}$$

- $N^{(1)}(0) = P_1 > P_2 = N^{(2)}(0)$

  & $N^{(1)}, N^{(2)}$ satisfy (I)

  $\Rightarrow N^{(1)}(t) > N^{(2)}(t)$

- $P=0$, $N(t)=0$

  $P=r/a$, $N(t)=r/a$ (Ex)

unstable

$N(t)$

$N_2 = N^{(2)}(0)$ T.

ity ⊣

$N^{(2)}(t)$

$= 0$

$= r/a$ (Ex)



unstable equilibrium

$N(t) = 0$

stable equilibrium

$N(t) = r/a$

$\Rightarrow$

$x(t)$

• $N = 0$

Ex.

$x(t$