Recall :-

$x : [0, \alpha) \longrightarrow \mathbb{R}$

Solves

(IVP) $\quad \frac{dx}{dt} = f(t, x(t)) \quad (*)$

$x(0) = \alpha$

$f : \mathbb{R}_+ \times \mathbb{R} \longrightarrow \mathbb{R}$

$$x(t) = x(0) + \int_0^t f(s, x(s))\, ds \quad (**)$$

Theorem :- Unif in t $|f(t,x) - f(t,y)| \leq c|x-y| \Rightarrow (**)$ has a unique solution

- $f(t,x) = \frac{g(t)}{h(x)}$

$$\int_{x(0)}^{x(t)} h(y)\, dy = \int_0^t g(s)\, ds$$

- $f(t,x) = xp + q$

- integrating factor

$f(t,x) = xp + qx^m$

$m \in \mathbb{R}$

- $z(t) = (x(t))^{\alpha}$

choose α to solve

- $f(t,x) = \dfrac{g(t)}{h(x)}$

$\int_{x(0)}^{x(t)} h(y)\,dy = \int_0^t g(s)\,ds$

$(t, x(t))$ (*)

α

- integrating factor

$f(t,x) = xp + q$

$f(t,x) = xp + qx^m$

$m \in \mathbb{R}$

$s, x(s))\,ds$ (*)

- $z(t) = (x(t))^{\alpha}$

choose α to solve

$|) | \le C|x-y| \Rightarrow$ (*) has a unique solution

- $f(t,x) = 3x^{2/3}$

$x(0) = 0$

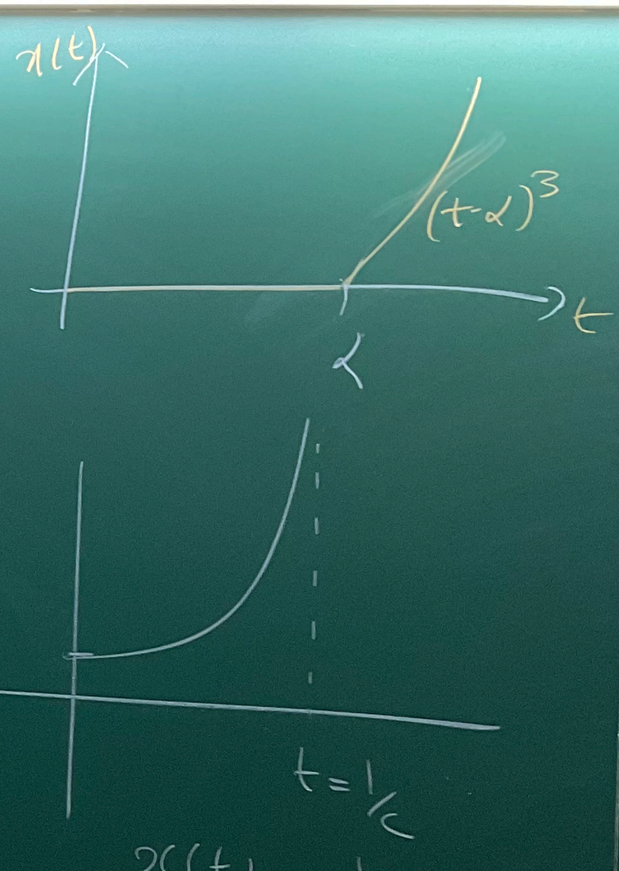
infinitely many solutions

- $f(t,x) = x^2$

$x(0) = C$

$(C > 0)$

finite time blow-up



- $g(t) \le \alpha + \beta \int_0^t g(s)\,ds \Rightarrow g(t) \le \alpha e^{\beta t}$

2nd order

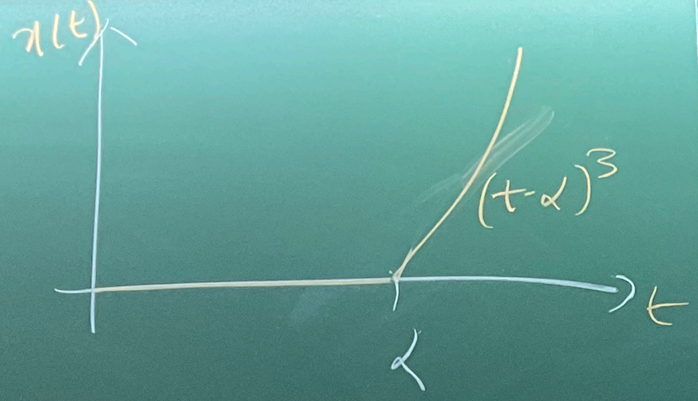
let $P, q,$

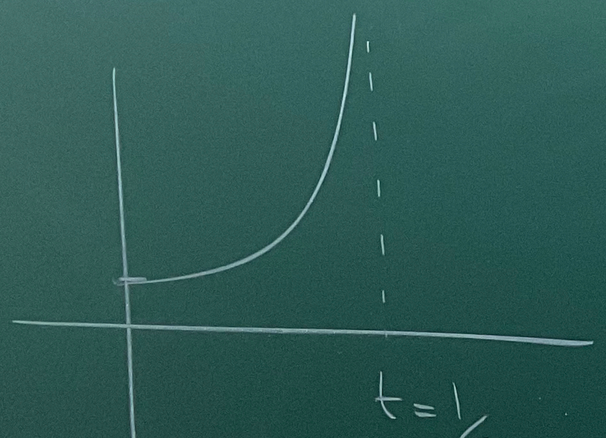
Scalar 2nd

$x : [0, \infty)$

(+) $\left\{ \dfrac{d^2x}{dt^2} + \right.$

General form





$$x(t) = \frac{1}{c-t}$$

$$\Rightarrow q(t) \leq \alpha e^{\beta t}$$

2nd order linear ODE's

Let $p, q, r : [0, \infty) \to \mathbb{R}$

Scalar 2nd order linear ODE

$x : [0, \infty) \to \mathbb{R}$

$$\oplus \begin{cases} \dfrac{d^2x}{dt^2} + p(t)\dfrac{dx}{dt} + q(t)x(t) + r(t) = 0 \\ x(0) = \alpha \end{cases}$$

General form of 2nd order ODE

$$\frac{d^2x}{dt^2} = g\left(t, x(t), x'(t)\right)$$

Plan:

- $r(t) \equiv 0$
- Higher dimensional 1st order ODE

- General case

$r(\cdot) \neq 0$

Homo

$V = \{ x$

where A

Theorem

V i

Σ dim

Homogeneous 2nd order linear ODE ($r=0$)

$$V = \left\{ \underset{\sim}{x} : [0,\infty) \to \mathbb{R}^2 \;\middle|\; \underset{\sim}{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ & } \frac{d\underset{\sim}{x}}{dt} = A\,\underset{\sim}{x} \right\}$$

where $A(t) = \begin{bmatrix} 0 & 1 \\ -q(t) & p(t) \end{bmatrix}$; $q, p : [0,\infty) \to \mathbb{R}$

Theorem :-

V is a vector space over $\mathbb{R}$ & $\dim(V) = 2$

$\frac{d^2x}{dt^2} + p\frac{dx}{dt} + qx = 0$ has two linearly independent solutions

(+) Higher dimensional 1st order linear ODE

$x_1(t) = x(t)$

$x_2(t) = \frac{d}{dt}x(t)$

$x_i : [0,\infty) \to \mathbb{R}$, $i = 1,2$

$\underset{\sim}{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

(+) (=) ① $\frac{dx_1}{dt} = x_2$ & ② $\frac{dx_2}{dt} + p x_2 + q x_1 + r = 0$

① (=) ② $\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -q & -p \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -r \end{bmatrix}$

2-dimensional 1st order linear ODE $\frac{d\underset{\sim}{x}}{dt} = A\,\underset{\sim}{x} + b$ $\quad A = \begin{bmatrix} 0 & 1 \\ -p & -q \end{bmatrix}$ & $b = \begin{bmatrix} 0 \\ -r \end{bmatrix}$

Constant case: $p(t)=p$; $q(t)=q$; $r(t)\equiv 0$

$$\boxed{\frac{d^2x}{dt^2} + p\frac{dx}{dt} + qx = 0} \quad -(+)$$

If $\lambda \in \mathbb{R}$, & $x(t) = e^{\lambda t}$ solves (+)

$$\lambda^2 x + p\lambda x + qx = 0 \quad \forall t \geq 0$$

$$x[\lambda^2 + p\lambda + q] = 0 \quad \forall t \geq 0$$

$$\lambda^2 + p\lambda + q = 0 \iff \begin{cases} (i)\ p^2 - 4q > 0, & \lambda_1 \& \lambda_2 \in \mathbb{R} \\ (ii)\ p^2 - 4q < 0, & \lambda_1, \lambda_2 \in \mathbb{C} \end{cases}$$

(iii) $p^2 - 4q = 0$, $-\frac{p}{2}$ is a ro

(i) $\lambda_1 = -\frac{p + \sqrt{p^2 - 4q}}{2}$

Two linearly independe

$x_1(t) = e^{\lambda_1 t}$

(ii) $p^2 - 4q < 0$

say $\lambda_1 = a + ib$

$$\boxed{e^{\lambda_1 t} = e^{at}(\cos bt +}$$

$= p$; $q(t) = q$; $x(t) \equiv 0$

$\ddot{x} = 0$ — ⊕

$e^{\lambda t}$ solve ⊕

$q x = 0 \quad \forall t \geq 0$

$= 0 \quad \forall t \geq 0$

$\Leftrightarrow$ (i) $p^2 - 4q > 0$, $\lambda_1 \& \lambda_2 \in \mathbb{R}$
(ii) $p^2 - 4q < 0$, $\lambda_1, \lambda_2 \in \mathbb{C}$

(iii) $p^2 - 4q = 0$, $-\frac{p}{2}$ is a real root

(i) $\lambda_1 = -\frac{p + \sqrt{p^2 - 4q}}{2}$ & $\lambda_2 = -\frac{p - \sqrt{p^2 - 4q}}{2}$

Two linearly independent solution are

$x_1(t) = e^{\lambda_1 t}$ $\quad x_2(t) = e^{\lambda_2 t}$

(ii) $p^2 - 4q < 0$

say $\lambda_1 = a + ib$ $\quad \lambda_2 = a - ib$

$$e^{\lambda_1 t} = e^{at}(\cos bt + i \sin bt)$$

$$e^{\lambda_2 t} = e^{at}$$

Ex

$\Rightarrow$ $x_1(t) =$

$x_2(t) =$

(iii) $p^2 - 4q = 0$

$x_1(t) = e$

DIVERSIO

Theorem:- let p, q

Suppose $x(\cdot)$ s

$$e^{\lambda_2 t} = e^{at}[\cos(bt) - i\sin(bt)]$$

Ex ⟹ $x_1(t) = e^{at}\cos(bt)$ are two l.i solutions.

$x_2(t) = e^{at}\sin(bt)$

(iii) $p^2 - 4q = 0$

$x_1(t) = e^{-p/2 t}$ is one solution.

DIVERSION

Theorem:- let p, q be continuous functions

Suppose $x(\cdot)$ is a solution

$\lambda_2 = \frac{-p - \sqrt{p^2 - 4q}}{2}$

nt solutions are

$x_2(t) = e^{\lambda_2 t}$

$\lambda_2 = a - ib$

$+ i(\sin bt)$

$\ddot{x} + p(t)$

& $x(t)$

Then

$x_2(t) = x$

is a solution

linearly ind

two l.i
solutions.

olution

ctions

$$\textcircled{+}\quad \frac{d^2x}{dt^2} + p(t)\frac{dx(t)}{dt} + q(t)x(t) + r(t) = 0$$

$\exists\ x(t) \neq 0 \quad \forall t \geq 0$

Then

$$x_2(t) = x(t)\int_0^t \frac{e^{-\int_0^s p(u)du}}{x(s)^2}\,ds$$

is a solution to $\textcircled{+}$ & is linearly independent of $x(\cdot)$

Back to Case (iii); Using Theorem

$$x_2(t) = e^{-\frac{pt}{2}}\int_0^t \frac{e^{-ps}}{e^{-ps}}\,ds$$

$$= e^{-\frac{pt}{2}}[t]$$

Facts

(I) $P, q, r \equiv$ Continuous functions

(+) $\dfrac{d^2x}{dt^2} + p(t)\dfrac{dx}{dt} + q(t)x(t) + r(t) = 0$

Suppose $\hat{x}(\cdot)$ is a particular solution to (+). Then any general solution to (+) will be of the form

$$x(t) = \hat{x}(t) + x_1(t) + x_2(t)$$

where $x_1(\cdot)$ and $x_2(\cdot)$ are linearly indep. soln.

(II)

Solutions of

(I) $\frac{d^2x}{dt^2} + p(t)\frac{dx}{dt} + q(t)\,x(t) = 0$

(II) Let $x_1(\cdot)$, $x_2(\cdot)$ be two linearly independent solutions of

(I) Then the

Wronskian w.r.t $x_1(\cdot)$ & $x_2(\cdot)$ is

$$W(t) = \det\begin{pmatrix} x_1(t) & x_2(t) \\ \frac{dx_1(t)}{dt} & \frac{dx_2(t)}{dt} \end{pmatrix}$$

then

$$W(t) = W(0)\, e^{-\int_0^t p(s)\,ds}$$

$x_2(\cdot)$

(III) Same set up as in (II)

Then the general solution to (†)

$$x(t) = c_1 x_1(t) + c_2 x_2(t) - \int_0^t \frac{[x_2(t)x_1(s) - x_1(t)x_2(s)]}{W(s)} r(s)\,ds$$

where c_1 & c_2 are constants & $W(\cdot)$ is wronskian