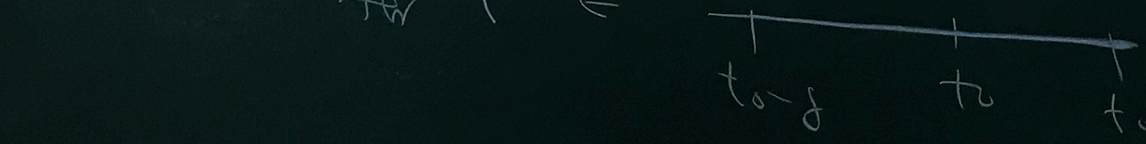
Power Series Solution

$f: \mathbb{R} \longrightarrow \mathbb{R}$ is real analytic at t_0

if $f(t) = \sum_{j=0}^{\infty} a_j (t-t_0)^j$

for $t \in$ 

$t \in (t_0-\delta, t_0+\delta)$

for some $\delta > 0$.

$\frac{f(t+h)-f(t)}{h} \to f'(t)$ as $h \to 0$

$(f_1+f_2+\cdots+f_n)' = f_1'+f_2'+\cdots+f_n'$

If

$T_n =$

Series

$$\sum_{n=1}^{\infty} b_n = b_1 + b_2 + b_3 + \cdots$$

$$T_n = \sum_{k=1}^{n} b_k \qquad \forall n \geq 1$$

If $n \to \infty$ then $T_n \longrightarrow b$ $\quad b \in \mathbb{R}$

$$b = \sum_{k=1}^{\infty} b_k$$

$\forall \varepsilon > 0 \ \exists N$

$T_n \ \forall n \geq N$

$b-\varepsilon \quad b \quad b+\varepsilon$

Theorem:

Let $p(\cdot), q(\cdot)$ be real analytic at $t_0 \geq 0$. Let $x : [0, \infty) \to \mathbb{R}$

Solve

$$\frac{d^2x}{dt^2} + p(t)\frac{dx}{dt} + q(t)x(t) + r(t) = 0$$

$$\frac{dx}{dt} + q\,x = 0 \qquad \forall t > 0$$

$$x(t_0) = x_0$$

Then $x(\cdot)$ is real analytic at t_0.

$(t) = 0$

Example :-

$$\frac{d^2x}{dt^2} + t\frac{dx}{dt} + x = 0 \qquad \forall t > 0$$

$p(t) = t$, $r(t) = 0$, $q(t) = 1$

$[t_0 = 0,\ a_1 = 1,\ a_k = 0\ \forall k \neq 1]$

Given the result, to find a solution

Set $\boxed{x(t) = \sum_{k=0}^{\infty} a_k t^k}$

Justify ; Rudin: P.M.A.

Interchange of limits

$$\frac{dx}{dt} = \frac{d}{dt}\left(\sum_{k=0}^{\infty} a_k t^k\right) = \sum_{k=0}^{\infty} \frac{d}{dt}\left(a_k t^k\right)$$

$$\frac{dx}{dt} = \boxed{\sum_{k=1}^{\infty} k\, a_k t^{k-1}}$$

$$\frac{d^2x}{dt} = \boxed{\sum_{k=2}^{\infty} k(k-1)\, a_k t^{k-2}}$$

①, ②, ③ into ✱

$$\Rightarrow \sum_{k=2}^{\infty} k(k-1)\,a_k t^{k-2} + t\sum_{k=1}^{\infty} k\,a_k t^{k-1} + \sum_{k=0}^{\infty} a_k t^k = 0 \qquad \forall t>0$$

$$\Rightarrow \sum_{m=0}^{\infty} (m+2)(m+1)\,a_{m+2} t^m + \sum_{k=0}^{\infty} k\,a_k t^k + \sum_{k=0}^{\infty} a_k t^k = 0 \qquad \forall t>0$$

$$\Rightarrow \sum_{m=0}^{\infty} \underbrace{\left[(m+2)(m+1)\,a_{m+2} + m\,a_m + a_m\right]}\, t^m = 0 \qquad \forall t>0$$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$\Rightarrow (m+2)(m+1)a_{m+2} + m a_m + a_m = 0 \quad \forall m \geq 0$$

$$\Rightarrow (m+2)(m+1)a_{m+2} = -(m+1)a_m \quad \forall m \geq 0$$

$$\Rightarrow a_{m+2} = -\frac{1}{m+2} a_m \quad \forall m \geq 0$$

Ex $a_{2k} = \dfrac{(-1)^k a_0}{2 \cdot 4 \cdots 2k} = \dfrac{(-1)^k a_0}{2^k k!}$, $k = 0, 1, 2, \ldots$

$a_{2k+1} = \dfrac{(-1)^k a_1}{1 \cdot 3 \cdots (2k+1)} = \dfrac{(-1)^k a_1 \, 2^k k!}{(2k+1)!}$, $k = 0, 1, 2, \ldots$

$$x_1(t) = \sum_{k=0}^{\infty} a_{2k} t^{2k} = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{2^k k!} = e^{-t^2/2} \qquad a_0 = 1, \; a_1 = 0$$

$$x_2(t) = \sum_{k=0}^{\infty} a_{2k+1} t^{2k+1} = \sum_{k=0}^{\infty} \frac{(-1)^k 2^k k! \, t^{2k+1}}{(2k+1)!} \qquad a_1 = 1, \; a_0 = 0$$

; $k=0,1,2,\ldots$

$2^k k!$.

$k=0,1,2,\ldots$

$a_0=1,\ a_1=0$

$a_1=1\quad a_0=0$

Example :-

$$\frac{d^2x}{dt^2} + t\frac{dx}{dt} + x = 0 \quad (*) \qquad \forall t>0$$

A general solution : $x(t) = a\,x_1(t) + b\,x_2(t)$ for any $a,b\in\mathbb{R}$.

Given the result, to find a solution

Set $x(t) = \sum_{k=0}^{\infty} a_k t^k$ — (1)

$\sum_{k=0}^{n} q_k t^k \longrightarrow x(t)$ as $n\to\infty$

$\forall \varepsilon>0$

$\exists N$ $n\geq N$

$\sum_{k=0}^{n} q_k t^k$

$|x(t) - x(t)| < \varepsilon$

Justify ; Rudin: P.M.A.

Interchange of limits

$$\frac{dx}{dt} = \frac{d}{dt}\left(\sum_{k=0}^{\infty} a_k t^k\right) = \sum_{k=0}^{\infty} \frac{d}{dt}\left(a_k t^k\right)$$

$$\frac{dx}{dt} = \sum_{k=1}^{\infty} k a_k t^{k-1} \quad -(2)$$

$$\frac{d^2x}{dt^2} = \sum_{k=2}^{\infty} k(k-1) a_k t^{k-2} \quad -(3)$$