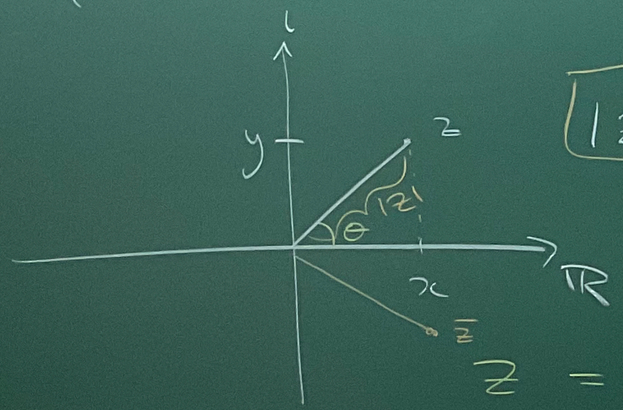
Complex Analysis

$$\mathbb{C} = \{ z = x + iy \mid x \in \mathbb{R},\ y \in \mathbb{R},\ "i = \sqrt{-1}" \}$$

Bombelli's Wild thought

$x^2 = mx + c$

$D = m^2 + 4c$



$$|z| = \sqrt{x^2 + y^2}$$

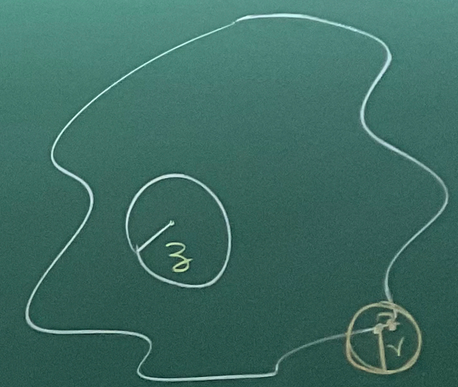
$$z = |z| e^{i\theta} = r(\cos\theta + i \sin\theta)$$

$|r = |z|$ — Polar coordinates

$$\bar{z} = x - iy$$

- $(x+iy)(a+ib) = (ax - by) + i(ay + bx)$
- $x+iy \oplus a+ib = (x+a) + i(y+b)$

$D \subseteq \mathbb{C}$



$z \in D$ is interior if $\exists r$ s.t. $B(z, r) \subseteq D$

$z \in \mathbb{C}$ a boundary point of D if $\forall r > 0$, $B(z, r) \cap D \neq \emptyset$ & $B(z, r)$

$D^\circ = \{ z \in D \mid z \text{ is interior point of } D \}$

$\partial D = \{ z \in \mathbb{C} \mid z \text{ is a boundary point of } D \}$

- $D = D^\circ$ then D is open
- D^c — open then D is closed

- D is connected

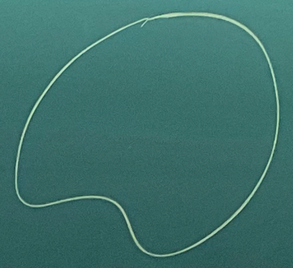
eg  , 

$\Rightarrow$ D cannot be a disjoint union of two open sets

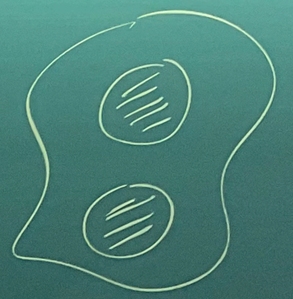
$\forall\, \gamma > 0$

$B(z,\gamma) \cap D^c \neq \emptyset$

D is Simply connected



D



if D is connected & "any curve in D can be shrinked to a point inside D"

Ω - domain in $\mathbb{C}$ → open & simply connected

$f : \Omega \to \mathbb{C}$, f is a continuous at $z_0 \in \Omega$ if $\lim_{z \to z_0} f(z) = f(z_0)$

$\forall\, \varepsilon > 0, \exists\, \delta >$

- f is h

If

converge to a $M \in \mathbb{C}$ denoted by $f'(z_0)$

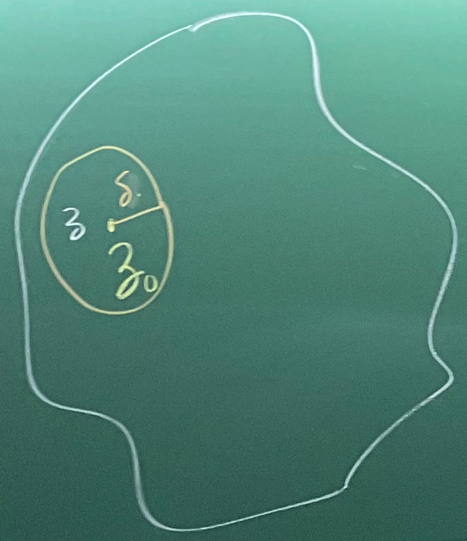
as $h \to 0$.

$$\lim_{\substack{h \to 0 \\ h \in \mathbb{C}}} \frac{f(z_0+h) - f(z_0)}{h} := f'(z_0)$$

Eg: $f(z) = z$ ✓ holomorphic

$f(z) = \bar{z}$ ✗ non-holomorphic

ie



$\forall \varepsilon > 0, \exists \delta > 0 \; |f(z) - f(z_0)| < \varepsilon$

Whenever $|z - z_0| < \delta$

- f is holomorphic at $z_0 \in \Omega$

If $\dfrac{f(z_0+h) - f(z_0)}{h}$

- f, g holomorphic

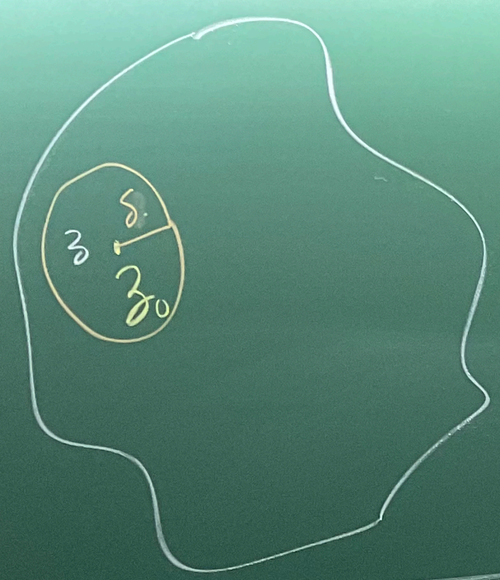
$(f+g)' = f' + g'$; $(fg)' = f'g + fg'$

(Additive) (Product rule)

$(f \circ g(z))' = f'(g(z))\, g'(z)$

(Chain rule)

i.e



Ω

$\forall \varepsilon > 0, \exists \delta > 0 \mid |f(z) - f(z_0)| < \varepsilon$ whenever $|z - z_0| < \delta$

Converges to a $M \in \mathbb{C}$ denoted by $f'(z_0)$ as $h \to 0$.

$$\lim_{\substack{h \to 0 \\ h \in \mathbb{C}}} \frac{f(z_0+h) - f(z_0)}{h} := f'(z_0)$$

Eg:

$P_n(z) = \sum_{k=0}^{n} a_k z^n$ — holomorphic

$f(z) = z$ ✓ holomorphic

$f(z) = \bar{z}$ × non-holomorphic

- f is holomorphic at $z_0 \in \Omega$ if $\frac{f(z_0+h) - f(z_0)}{h}$

- f, g holomorphic
  $(f+g)' = f' + g'$; $(fg)' = f'g + fg'$
  (Additive) (Product rule)

$(f \circ g)'(z) = f'(g(z)) g'(z)$
(chain rule)

Cauchy-Riemann Equations

$f: \Omega \to \mathbb{C}$ & f is holomorphic on Ω & $f = u + iv$

then $\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \;\&\; \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$

Proof:- $f(z) = u(x,y) + iv(x,y)$

When $z = x + iy$

$h = h_1 + ih_2$; $z+h = (x+h_1) + i(y+h_2)$

Mean Value Theorem

$(*_1)$ $u(x+h_1, y+h_2) - u(x,y)$

$(*_2)$ $v(x+h_1, y+h_2) - v(x,y)$

$$\frac{f(z+h) - f(z)}{h} = \frac{u(x+h_1, y+h_2) + iv(x+h_1, y+h_2)}{h_1 + ih_2}$$

$$= \frac{(*_1) + i(*_2)}{h_1 + ih_2}$$

Mean Value Theorem

(*1) $u(x+h_1, y+h_2) - u(x,y) = h_1 \frac{\partial u}{\partial x}(\ ,\ ) + h_2 \frac{\partial u}{\partial y}(\ ,\ ) + |h|\, q_1(h)$

(*2) $v(x+h_1, y+h_2) - v(x,y) = h_1 \frac{\partial v}{\partial x}(\ ,\ ) + h_2 \frac{\partial v}{\partial y}(\ ,\ ) + |h|\, q_2(h)$

where $q_i(h) \to 0$ as $h \to 0$ for $i = 1, 2$

$$\frac{f(z+h) - f(z)}{h} = \frac{u(x+h_1, y+h_2) + i\,v(x+h_1, y+h_2) - u(x,y) - i\,v(x,y)}{h_1 + i h_2}$$

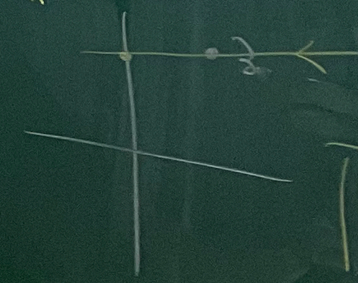
$$= \frac{(*_1) + i\,(*_2)}{h_1 + i h_2}$$

$$\frac{f(z+h) - f(z)}{h}$$

know f is

$\Rightarrow \lim_{\substack{h \to 0 \\ h \in \mathbb{C}}}$

$h = h_1$

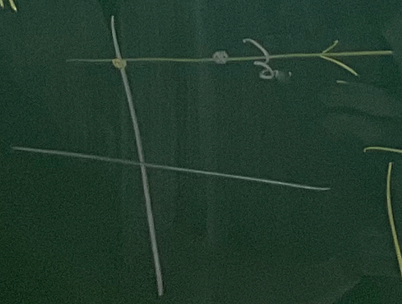


$$\frac{f(z+h)-f(z)}{h} = \frac{h_1\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)+h_2\left(\frac{\partial u}{\partial y}+i\frac{\partial v}{\partial y}\right)+|h|q_1(h)+i|h|q_2(h)}{h_1+ih_2} \quad -(\dagger)$$

Know f is holomorphic

$$\Rightarrow \lim_{\substack{h\to 0\\ h\in\mathbb{C}}} \frac{f(z+h)-f(z)}{h} = f'(z_0)$$

$h=h_1$



$$\frac{f(z_0+h)-f(z)}{h} = \frac{h_1\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)+|h_1|q_1(h)+|h_1|q_2(h)}{h} \quad (\dagger)$$

$h\to 0$

$$f'(z_0)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$$

h_1

$h=ih_2$

$h\to 0$

From ① & ②

$$\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$$

$q_1(h) + i|h| q_2(h)$ — (+)

$h = ih_2$

$$\frac{f(z_0+h) - f(z_0)}{h} = \frac{h_2\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right) + |h| q_1(h) + i|h| q_2(h)}{ih_2}$$

$h \to 0$

$$f'(z_0) = \frac{\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}}{i} \quad \text{②}$$

From ① & ②

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}}{i}$$

$$= -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$\Rightarrow$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad \square$$

$|h_1| q_1(h) + |h_1| q_2(h)$ — (+)