Recall:

$\gamma : [a,b] \longrightarrow \mathbb{C}$

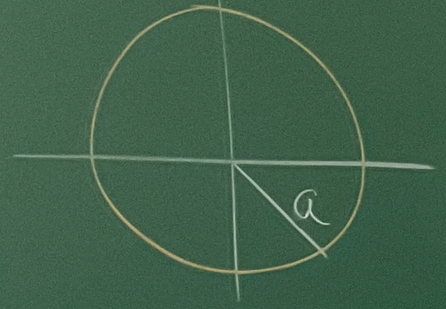
- (parametrised curve)
- Smooth curve if
  $\gamma'(\cdot)$ exist

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt$$

continuous function

Example

$\gamma \equiv$ Circle of radius $a > 0$ around the origin in $\mathbb{C}$



$f(z) = z^n \qquad n \geq 1$

$\gamma : [0, 2\pi] \to \mathbb{C}$

$\gamma(\theta) = a e^{i\theta} = a(\cos\theta + i \sin\theta)$

$\gamma'(\theta) = a(-\sin\theta + i\cos\theta)$

$= ia(\cos\theta + i\sin\theta) = iae^{i\theta}$

$$\int_\gamma f(z)\,dz = \int_0^{2\pi} f(\gamma(\theta))\,\gamma'(\theta)\,d\theta$$

$$= \int_0^{2\pi} a^n e^{in\theta}\, ie^{i\theta} a\,d\theta$$

$$= i a^{n+1} \int_0^{2\pi} e^{i(n+1)\theta} d\theta$$

$$= i a^{n+1} \left. \frac{e^{i(n+1)\theta}}{i(n+1)} \right|_0^{2\pi}$$

$$= 0$$

$$\oint_\gamma z^n dz = 0, \quad \forall n \geq 1$$

z^n - $n \geq 1$ is analytic inside γ

$$f(z) = \begin{cases} \frac{1}{z}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

Not analytic inside the space enclosed by γ.

$$\oint_\gamma f(z) dz = \int_0^{2\pi} a^{-1} e^{-i\theta} \, i e^{i\theta} a \, d\theta$$

$$= i 2\pi$$

$$\oint_\gamma \frac{1}{2} dz = 2\pi i$$

$F' = f$

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Cauchy / Goursat [Without Proof]

- $\gamma : [a,b] \to \mathbb{C}$ is a closed (smooth) curve
  (i.e. $\gamma(a) = \gamma(b) \in \mathbb{C}$)
  & simple (i.e. if $\gamma(t_1) = \gamma(t_2)$ then $t_1 = t_2$ or $t_1 = a$ & $t_2 = b$)
- $f : \mathbb{C} \to \mathbb{C}$ & f is analytic/holomorphic in Ω
- $\text{Range}(\gamma) \subseteq \Omega$.
- "all points surrounded" by $\gamma \subseteq \Omega$.

then $$\int_\gamma f(z)\,dz = 0.$$

$\gamma: [0, 2\pi) \to \mathbb{C}$; $\gamma(\theta) = z_0 + a e^{\theta}$

- $z_0 \in \mathbb{C}$ $\quad \gamma \equiv$ circle around z_0 of radius a

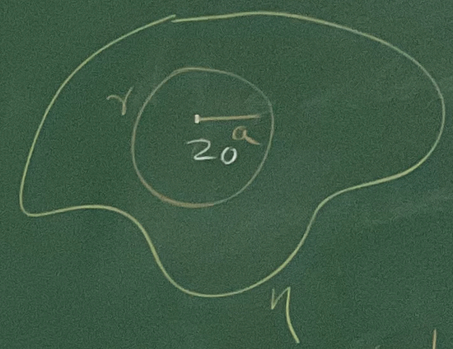
- $f: \mathbb{C} \to \mathbb{C}$ is analytic function

$$\int_\gamma \frac{f(z)}{z - z_0}\, dz = \int_0^{2\pi} \frac{f(z_0 + a e^{i\theta})}{a e^{i\theta}}\, i a e^{i\theta}\, d\theta$$

$$= i \int_0^{2\pi} f(z_0 + a e^{i\theta})\, d\theta \quad \text{└ (1)}$$

$f: \mathbb{C} \to \mathbb{C}$ is analytic

Fact:- η - is any closed simple curve containg z_0



For some $a > 0$ γ - circle of radius a around z_0 is enclosed by η

$$\int_\eta \frac{f(z)}{z - z_0}\, dz = \int_\gamma \frac{f(z)}{z - z_0}\, dz$$ Without Proof

└ (2)

From ① and ②

$$\int_{\eta} \frac{f(z)}{z-z_0}\,dz = i\int_0^{2\pi} f(z_0+a e^{i\theta})\,d\theta$$

for $a>0$ such that $\gamma'' \subseteq \eta''$ (small enough)

let $a \to 0$,

$$\lim_{a\to 0} f(z_0+a e^{i\theta}) = f(z_0)$$

η - simple closed curve containing z_0

$$\Rightarrow \int_{\eta} \frac{f(z)}{z-z_0}\,dz = i f(z_0)\int_0^{2\pi} 1\,d\theta$$

(Ex)

$$= i f(z_0)\, 2\pi$$

Cauchy Integral formula

f - analytic in $\mathbb{C}$ & η - any closed curve containing z_0

$$\frac{1}{2\pi i}\int_{\eta} \frac{f(z)}{z-z_0}\,dz = f(z_0)$$

$z_0 \in \mathbb{C}$, η is a closed curve containing z_0

$f: \mathbb{C} \to \mathbb{C}$ is analytic

((IF) $$f(z_0) = \frac{1}{2\pi i} \int_\eta \frac{f(z)}{z - z_0}\, dz$$

ε small & $\varepsilon > 0$

$$\frac{f(z_0+\varepsilon) - f(z_0)}{\varepsilon} = \frac{1}{2\pi i} \int_\eta f(z) \frac{1}{\varepsilon} \left[\frac{1}{z-(z_0+\varepsilon)} - \frac{1}{(z-z_0)} \right]$$

$\varepsilon \to 0$

(Ex) $$f'(z_0) = \frac{1}{2\pi i} \int_\eta \frac{f(z)}{(z-z_0)^2}\, dz$$

Rep

$n \geq 1$

Repeat the argument

$$f''(z_0) = \frac{1}{2\pi i}\, 2 \int_\eta \frac{f(z)}{(z-z_0)^3}\, dz$$

$n \geq 1$

$$f^{(n)}(z_0) = \frac{1}{2\pi i}\, n! \int_\eta \frac{f(z)}{(z-z_0)^{n+1}}\, dz$$

$\longrightarrow$ f is differentiable infinitely many times ③

$$f(z_0) = \frac{1}{2\pi i} \int_\eta \frac{f(z)}{z-z_0}\, dz$$

$a \in \mathbb{C}$

$$= \frac{1}{2\pi i} \int_\eta \frac{f(z)}{(z-a)-(z_0-a)}\, dz$$

$$= \frac{1}{2\pi i} \int_\eta \frac{f(z)}{(z-a)\left(1-\frac{z_0-a}{z-a}\right)}$$

Fact: $q \in \mathbb{C}$ $|q|<1$ $\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}$

(Ex)
Choose $a \in \mathbb{C}$ such that

$$\frac{|z_0 - a|}{|z - a|} < 1 \quad \forall z \in \eta.$$

Using Fact

$$f(z_0) = \frac{1}{2\pi i} \int_\eta \frac{f(z)}{z-a} \left[\sum_{n=0}^{\infty} \left(\frac{z_0 - a}{z-a} \right)^n \right] dz$$

$$(Ex) = \sum_{n=0}^{\infty} (z_0 - a)^n \cdot \frac{1}{2\pi i} \int_\eta \frac{f(z)}{(z-a)^{n+1}} dz$$

From (3)

$$= \sum_{n=0}^{\infty} (z_0 - a)^n \frac{f^{(n)}(a)}{n!}$$

$\Rightarrow$ (Ex) f - admits Taylor series

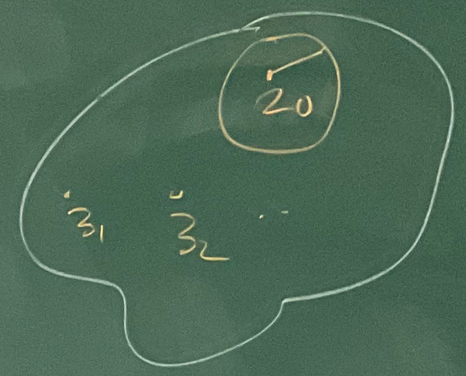
$$f(z_0) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (z_0 - a)^n$$

$z_0, a \in \Omega$

bounded

Ω - Simply connected open set

$f : \Omega \to \mathbb{C}$ analytic



Zeros of f are isolated

Result - Suppose $z_i \in \Omega$ $f(z_i) = 0$ $\forall i \geq 1$

and $z_i \to z \in \Omega$ as $i \to \infty$

(z_i accumulate around z) $(f(z_i) \to f(z) \Rightarrow f(z) = 0)$

$\Rightarrow f \equiv 0$

• $z_0 \in \Omega$ & $f(z_0) = 0$, has a finite order

Then one can take a Ball U around z_0 in Ω

& find $n \geq 1$ (Unique), $g: U \to \mathbb{C}$

$g(z_0) \neq 0$

g - analytic

such that $f: U \to \mathbb{C}$

$$f(z) = (z - z_0)^n g(z)$$

$$f(z_0) = \frac{1}{2\pi i} \int_\eta \frac{f(z)}{z - z_0}\, dz$$

$a \in \mathbb{C}$

$$= \frac{1}{2\pi i} \int_\eta \frac{f(z)}{(z-a) - (z_0 - a)}\, dz$$

$$= \frac{1}{2\pi i} \int_\eta \frac{f(z)}{(z-a)\left(1 - \frac{z_0 - a}{z - a}\right)}$$

Fact: $q \in \mathbb{C}$ $|q| < 1$ $\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}$

(Ex)

Choose

Using

$f(z_0) =$

(Ex) =

(Ex)
Choose $a \in \mathbb{C}$ such that

$$\frac{|z_0 - a|}{|z - a|} < 1 \quad \forall z \in \eta.$$

Using Fact

$$f(z_0) = \frac{1}{2\pi i} \int_\eta \frac{f(z)}{z-a} \left[\sum_{n=0}^{\infty} \left(\frac{z_0 - a}{z-a} \right)^n \right] dz$$

$$(Ex) = \sum_{n=0}^{\infty} (z_0 - a)^n \cdot \frac{1}{2\pi i} \int_\eta \frac{f(z)}{(z-a)^{n+1}} dz$$

From (3)

$$= \sum_{n=0}^{\infty} (z_0 - a)^n \frac{f^{(n)}(a)}{n!}$$

$\Rightarrow$ (Ex) f - admits Taylor series

$$f(z_0) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (z_0 - a)^n$$

$z_0, a \in \phi$