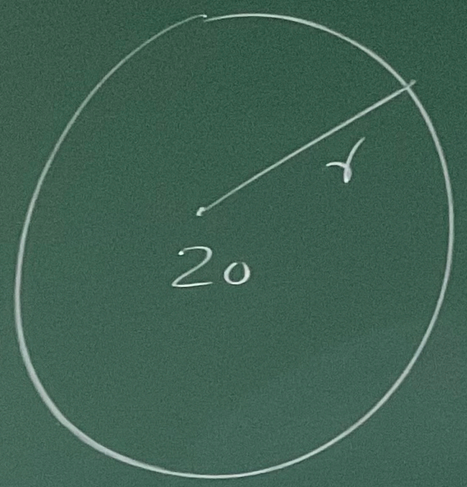
Residues
Branch - cuts



$f: \mathbb{C} \to \mathbb{C}$ and is analytic

on $B(z_0, r)$. $f(z_0) = 0$

Then $\exists\, k \geq 1$ (order of the zero)

$$f(z) = (z - z_0)^k \sum_{n=0}^{\infty} a_n (z - z_0)^{n-k}$$

$z \in B(z_0, r)$

? ← Convergence of the series

Singularities

$f: \mathbb{C} \to \mathbb{C}$ i

it is no

- Isolated sing

If f is an

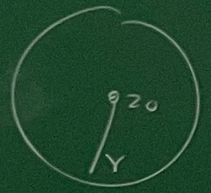
$z: 0 < |$

Singularities

$f: \mathbb{C} \to \mathbb{C}$ is singular at z_0 if it is not analytic at z_0

- **Isolated singularity** at z_0 if f is analytic at

$z: \; 0 < |z - z_0| < r$



anywhere of the series

Example. $f(z) = \begin{cases} \frac{1}{z} & z \neq 0 \\ 0 & z = 0 \end{cases}$ $0 < |z|$

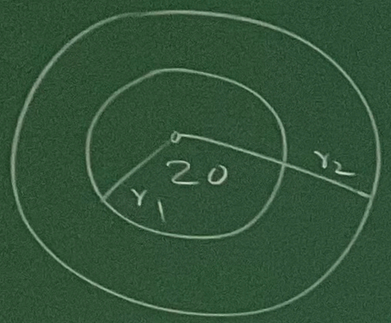
$z_0 = 0$ is an isolated singularity of f

$f(z) = \begin{cases} \log(z) & z \neq 0 \\ 0 & z \in (-\infty, 0] \end{cases}$

$z_0 = 0$ is a singularity & is not isolated

Laurent series

Suppose $f: \mathbb{C} \to \mathbb{C}$ & $z_0 \in \mathbb{C}$ analytic $\{z \in \mathbb{C} \mid r_1 < |z - z_0| < r_2\}$



Then

$$f(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} + \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

Example $f(z) = \begin{cases} \frac{1}{2} & z \neq 0 \\ 0 & z = 0 \end{cases}$, $z_0 = 0$

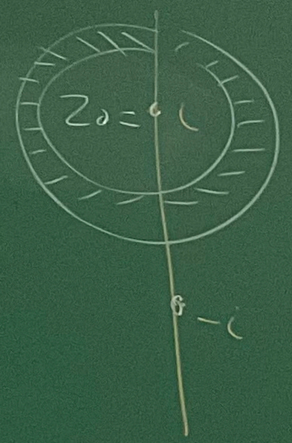
$b_1 = 1$ $\quad a_n = 0 \ \forall n \geq 0$

$b_n = 0 \ \forall n \geq 2$

$f(z) = \frac{1}{z}$ on $r_1 < |z| < r_2$

$\forall r_1, r_2 > 0$

- $f(z) = \frac{z}{z^2+1}$ Ex. $z = i$, $z = -i$ are singularities



$= \frac{z}{(z-i)(z+i)}$

$= \frac{1}{2} \frac{1}{z-i} + \frac{1}{2} \frac{1}{z+i}$

$$\frac{1}{2+i} = \frac{1}{2i+2-i} = \frac{1}{2i}\left(\frac{1}{1+\frac{2-i}{2i}}\right)$$

$$\left(\left|\frac{2-i}{2i}\right| < 1\right) \quad = \frac{1}{2i}\sum_{n=0}^{\infty}\left(-\left(\frac{2-i}{2i}\right)\right)^n$$

$$= \frac{1}{2i}\sum_{n=0}^{\infty}\left(\frac{-1}{2i}\right)^n (2-i)^n$$

$0 < |2-i| < 2$

$$f(2) = \frac{1}{2}\,\frac{1}{2-i} + \sum_{n=0}^{\infty}\frac{(-1)^n}{2(2i)^{n+1}}(2-i)^n$$

$f: \mathbb{C} \to \mathbb{C}$ has a isolated singularity at $z_0 \in \mathbb{C}$

Then $z_0 \in \mathbb{C}$ is referred to as a Pole

$\Rightarrow \exists r > 0$:

- f - analytic on $\{z \in \mathbb{C} \mid 0 < |z - z_0| < r\}$

$$f(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} + \sum_{n=1}^{\infty} a_n (z-z_0)^n$$

If there is a k such that $b_k \neq 0$ & $b_n = 0 \ \forall n > k$

then we say $z_0 \in \mathbb{C}$ is
a pole of order k

$k=1$, $z_0 \equiv$ simple pole

$b_n \neq 0$
$\forall n \geq 1$; $z_0 \equiv$ essential singularity

if $b_n = 0 \ \forall n \geq 1$ then
$z_0 \in \mathbb{C}$ is a removable
singularity: $f(z) = \begin{cases} z & z \neq 1 \\ 5 & z = 1 \end{cases}$

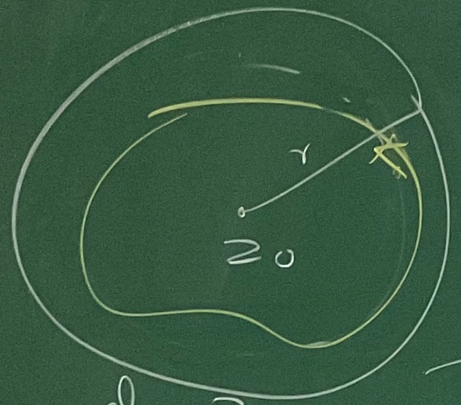
Residues

$f: \mathbb{C} \to \mathbb{C}$ has an isolated singularity
at $z_0 \in \mathbb{C}$ if $\exists r > 0$: $0 < |z - z_0| < r$

$$f(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} + \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

Then $\operatorname{Res}(f; z_0) := b_1$

- If γ is a simple closed curve – counter clockwise



$\gamma \in \{z : |z - z_0| < r\}$

around z_0. Then

$$\int_\gamma f(z)\,dz = 2\pi i \, b_1$$

- $f(z) = \dfrac{z}{z^2+1}$ as seen earlier

$$\operatorname{Res}(f, i) = \frac{1}{2}$$

- $f(z) = \dfrac{-1}{z(1-z)}$ $\quad 0 < |z| < 1$

$z_0 = 0$ $\quad f(z) = \dfrac{-1}{z}\left(\sum_{n=0}^{\infty} z^n\right)$

$\operatorname{Res}(f, 0) = -1$

$$= -\frac{1}{z} - 1 - \sum_{n=1}^{\infty} z^n$$

(2) Suppose

$g(z) = (z - z_0) f(z)$

if g is either

- analytic at z_0 OR

- f — simple pole at z_0

- removable singularity at z_0

$$\operatorname{Res}(f, z_0) = g(z_0)$$

$$= \lim_{z \to z_0} (z - z_0) f(z)$$

Example

$$f(z) = \frac{4 + z + z^2}{(z-2)(z-5)(z-6)(z-7)}$$

Poles $z, 5, 6, 7$

$$\text{Res}(f, 2) = (z-2) f(z) \Big|_{z=2}$$

$$= \frac{4+2+2^2}{(2-5)(2-6)(2-7)} = \frac{10}{-60} = \frac{-1}{6}$$

Cauchy Theorem for Residues

$f : \mathbb{C} \to \mathbb{C}$

If f has k isolated singularities $\{z_j \mid 1 \le j \le k\}$ & γ is a curve that surrounds $\{z_j \mid 1 \le j \le k\}$ counterclockwise

$$\int f(z)\,dz = 2\pi i \sum_{j=1}^{k} \operatorname{Res}(f, z_j)$$

logarithm in $\mathbb{C}$ $\equiv$ { Multivalued, Branch cuts, Quotient out

$z \in \mathbb{C}$ $\quad z \neq 0$ $\quad z = r e^{i\theta}$

$r = |z|$

(argument of z) $\theta \equiv \arg(z) \in I$ - half open interval of length 2π

ANY

choice: $\arg_I(z) \in I$

of argument of z

Example:

$z = -i$

$I = [0, 2\pi)$ $\quad \arg_I(z) = \frac{3\pi}{2}$

$I = (-\pi, \pi]$ $\quad \arg_I(z) = -\frac{\pi}{2}$

$I = (2\pi, 4\pi]$ $\quad \arg_I(z) = \frac{7\pi}{2}$

"$\arg_I(z)$" - has a "cut line" where the value jumps by 2π

$z \in \mathbb{C}$, $z \neq 0$

Find $w \in \mathbb{C}$ st $z = e^w$.

- only many solutions

$w = u + iv \Rightarrow z = (e^u)\, e^{iv}$

$u = \log|z|$

e^{iv} ← only many choices ← on choice of argument of z

(*) $\theta = \arg_I(z)$ for some I

We can take

$$w = u + iv$$

$$u = \log|z|$$

$$v = \theta + 2\pi k \quad k \in \mathbb{Z}$$

$$\log(z) = \log|v| + i(\arg_I(z)) + i2\pi k \quad k \in \mathbb{Z}$$

Example:-

$\arg_I(-3) = +\pi$

$$\log(-3) = \log(3) + i\pi + i2\pi k \quad k \in \mathbb{Z}$$

$$\log(5+2i) = \log(\sqrt{29}) + i\arctan\left(\frac{2}{5}\right) + i2\pi k \quad k \in \mathbb{Z}$$

A branch of $\log(z)$ will "jump" by $2\pi i$ as z crosses the cut line.

$$\log(e^z) = z + i(2\pi k) \quad \text{for some } k \in \mathbb{Z}$$

(Depends on Branch)

$w \in \mathbb{C}$

$$\log(zw) = \log z + \log w + i(2\pi k)$$

for some $k \in \mathbb{Z}$

For any branch of $\log(z)$

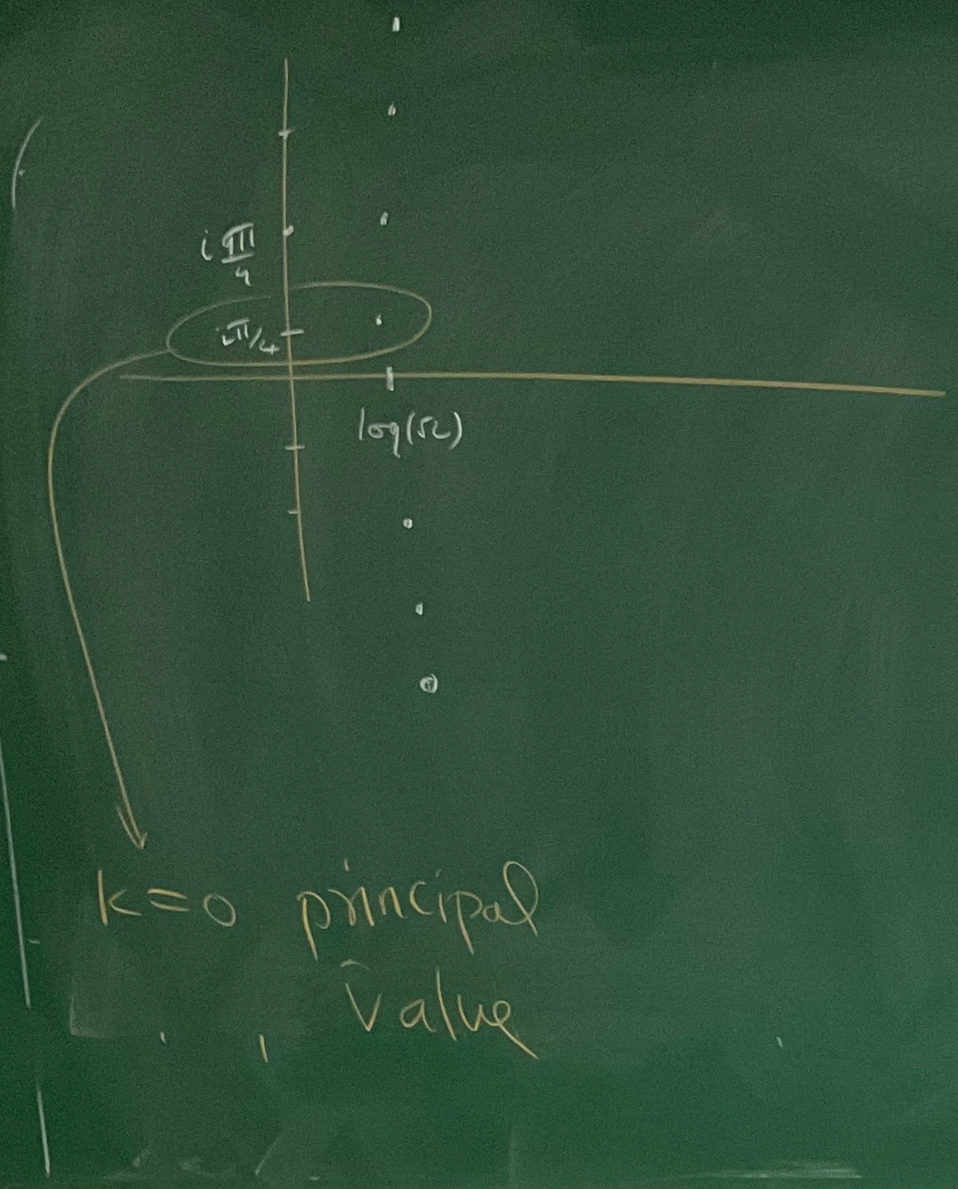
- $e^{\log(z)} = z$

Example

$z = 1 + i$

$$\log(1+i) = \log(\sqrt{2}) + i\frac{\pi}{4} + i2\pi k \quad k \in \mathbb{Z}$$

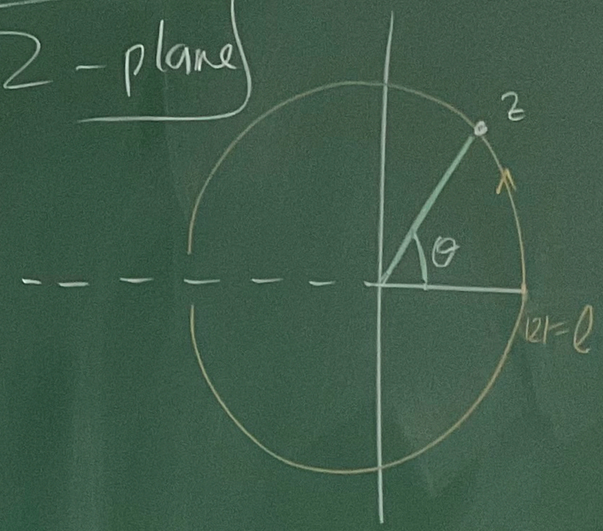
$I = (-\pi, \pi]$



k=0 principal value

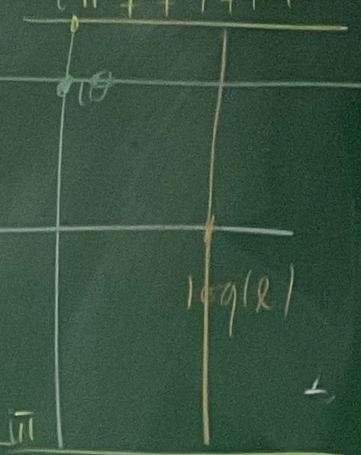
$W = \log(z)$

Z-plane



$G \mid (-\pi, 0] \to \mathbb{Z}$

$\log()$ → W-plane



$I = (-\pi, \pi]$ Branch $\log(z)$

$$\log_m(z) = \underbrace{\log z} + i 2\pi m \qquad m \in \mathbb{Z}$$

$$I = (-\pi, \pi)$$

